40 mins

Reconstructing topologies on monoids

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Monoids and inverse monoids

Semigroup	$*: S \times S \to S$ is associative
Monoid	Semigroup with identity
Inverse Semigroup	Semigroup with map $s \mapsto s^{-1}$ s.t. $ss^{-1}s = s$, $(s^{-1})^{-1} = s$, $ss^{-1}tt^{-1} = tt^{-1}ss^{-1}$
Inverse Monoid	Inverse semigroup with identity

semigroups $\rightleftharpoons$ monoids

Cayley's Theorem

Each monoid M embeds into the full transformation monoid M^M

Wagner-Preston Theorem

Each inverse monoid M embeds into the symmetric inverse monoid I_M.

Monoids	Endomorphisms
Inverse Monoids	Partial Automorphisms
Groups	Automorphisms

$\boxed{\text{Topological semigroup}} = \boxed{\text{semigroup } (S, *)} + \boxed{\text{space } (S, \tau)} + \boxed{\text{continuous } *: S \times S \to S}$

$\boxed{\text{Topological Inverse Semigroup}} = \boxed{\text{inverse semigroup } (S, *, ^{-1})} + \boxed{\text{space } (S, \tau)} + \boxed{\begin{array}{l}\text{continuous} \\ *: S \times S \to S \\ ^{-1}: S \to S\end{array}}$

(topological groups)

Well-known examples

$(\mathbb{R},+)$, $(\mathbb{R},\times)$, $(\mathbb{R},\max)$, $M_{n\times n}(\mathbb{R})$

replace $\mathbb{R}$ with $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$.

Profinite (semi)groups

$C(X,X)$ for a compact metric space X.

(compact-open topology)

discrete semigroups

Question

If M is a monoid, then how should we topologize M?

Which $\mathcal{T} \in P(P(M))$ is best?

What do we want?

Hausdorff

Second Countable

hereditary good

Polish

Compact

BCT good

(exclude discrete and indiscrete topologies)

Full Transformation Monoid $(\mathbb{N}^{\mathbb{N}}, \circ)$

Theorem (E, Jonušas, Mesyan, Mitchell, Morayne, Péresse)
There is a unique second countable Hausdorff semigroup topology on $(\mathbb{N}^{\mathbb{N}}, \circ)$.

Subbasis: $U_{x,y} := \{ f \in \mathbb{N}^{\mathbb{N}} \mid (x)f = y \}$ for $x, y \in \mathbb{N}$

Polish $\wedge$ $\neg$compact.

Endomorphisms of Relational Structures

Theorem (Folklore)
A submonoid M of $\mathbb{N}^{\mathbb{N}}$ is the endomorphism monoid of a relational structure if and only if M is closed

Theorem (Bodirsky, Junker)
If A and B are non-contractible ω-categorical structures, then

$\mathrm{End}(A)$ top iso $\mathrm{End}(B)$ $\Longleftrightarrow$ A and B positive existentially bi-interpretable

Topological submonoids of $\mathbb{N}^{\mathbb{N}}$

Theorem (E, Jonušas, Mesyan, Mitchell, Morayne, Péresse)
If S is a Hausdorff semigroup, then S topologically embeds into $\mathbb{N}^{\mathbb{N}}$ if and only if S has a subbasis consisting of the classes of a countable family of right congruences of countable index.

Theorem (Bardyla, E, Mitchell, Péresse)
If S is a compact semigroup then TFAE:

i) S embeds in $\mathbb{N}^{\mathbb{N}}$ as a topological semigroup

ii) S embeds in $\mathbb{N}^{\mathbb{N}}$ as a topological space

iii) S is metrizable and totally disconnected

Symmetric inverse monoid $I_{\mathbb{N}}$

Theorem (E, Jonušas, Mesyan, Mitchell, Morayne, Péresse)

There is a unique second countable Hausdorff inverse semigroup topology on $(I_{\mathbb{N}}, \circ)$.

Subbasis:
$$U_{x,y} := \left\{ f \in I_{\mathbb{N}} \;\middle|\; \begin{array}{l} x \in \operatorname{dom}(f) \\ (x)f = y \end{array} \right\}$$
$$V_x := \{ f \in I_{\mathbb{N}} \mid x \notin \operatorname{dom}(f) \}$$
$$W_x := \{ f \in I_{\mathbb{N}} \mid x \notin \operatorname{im}(f) \}$$
for $x, y \in \mathbb{N}$

Polish $\wedge$ $\neg$ compact.

Partial Automorphisms of Relational Structures

Theorem (Hampenberg Christenson)

An inverse submonoid M of $I_{\mathbb{N}}$ is the partial isomorphism monoid of a relational structure if and only if M is closed and contains all partial identity maps.

Theorem ?

If A and B are ?? ω-categorical structures, then

$\mathrm{Piso}(A)$ top iso $\mathrm{Piso}(B) \iff$???

Topological Inverse Submonoids of $I_{\mathbb{N}}$

Theorem (E, Jonušas, Mesyan, Mitchell, Morayne, Péresse)
If S is a Hausdorff inverse semigroup, then S topologically embeds in $I_{\mathbb{N}}$ if and only if S has a subbasis consisting of the classes and inverses of classes of a countable family of Wagner-Preston congruences.

Theorem (Bardyla, E, Mitchell, Péresse)
If S is a compact inverse semigroup then TFAE:

i) S embeds into $I_{\mathbb{N}}$ as a topological inverse semigroup
ii) S embeds into $I_{\mathbb{N}}$ as a topological space
iii) S is metrizable and totally disconnected

The Symmetric Group $\mathrm{Sym}(\mathbb{N})$

Theorem (Kechris, Rosendal)
There is a unique second countable Hausdorff group topology on $\mathrm{Sym}(\mathbb{N})$

Subbasis: $U_{x,y} := \{ f \in \mathrm{Sym}(\mathbb{N}) \mid (x)f = y \}$ $x, y \in \mathbb{N}$

Polish $\wedge$ $\neg$ compact.

Automorphisms of Relational Structures

Theorem (Folklore)
A subgroup G of $\mathrm{Sym}(\mathbb{N})$ is the automorphism group of a relational structure if and only if G is closed.

Theorem (Coquand)
If A and B are non-trivial ω-categorical structures then

$\mathrm{Aut}(A)$ top iso $\mathrm{Aut}(B)$ $\iff$ A and B bi-interpretable

Topological Subgroups of $\mathrm{Sym}(\mathbb{N})$

__Theorem__ (Becker, Kechris)

A Hausdorff topological group G embeds in $\mathrm{Sym}(\mathbb{N})$ iff 1_G has a countable nbhd basis of open subgroups.

(non-archimedean)

More examples

The following monoids admit a unique Polish topology:

1) (E, Jonušas, Mesyan, Mitchell, Morayne, Péresse)

- $P_{\mathbb{N}}$
- $C([0,1]^\omega)$
- $C(2^\omega)$

2) (E, Jonušas, Mitchell, Péresse, Pinsker) The endomorphism monoids of

- Random graph
- Random digraph
- Random poset
- $\bigsqcup_{i \in \omega} \mathbb{K}_n$

Standard Techniques (Lower Bounds)

- Fréchet-Markov / Hausdorff-Markov topologies
(intersection of all Fréchet/Hausdorff semigroup topologies)

- minimum T_1 shift-continuous topology

- Zariski topology

Closed $\longrightarrow$ $\{ s \in S \mid a_0 s a_1 s \dots a_{k-1} s a_k = b_0 s b_1 s \dots b_{l-1} s b_l \}$

- Inverse-Zariski topology

Closed $\longrightarrow$ $\{ s \in S \mid a_0 s^{\varepsilon_0} a_1 s^{\varepsilon_1} \dots a_{k-1} s^{\varepsilon_{k-1}} a_k = b_0 s^{\delta_0} b_1 s^{\delta_1} \dots b_{l-1} s^{\delta_{l-1}} b_l \}$

Theorem (E, Jonušas, Mesyan, Mitchell, Morayne, Péresse) + (Bardyla, E)

$$\begin{tikzcd}
& \boxed{\text{Hausdorff-Markov}} \arrow[dl, dash, "\subsetneq"'] \arrow[dr, dash, "\supsetneq"] & \\
\boxed{\text{Zariski}} \arrow[dr, dash, "\supsetneq"'] & & \boxed{\text{Fréchet-Markov}} \arrow[dl, dash, "\subsetneq"] \\
& \boxed{\text{Min } T_1} &
\end{tikzcd}$$

(no more containments can be added)

$I_{\mathbb{N}}$

$$\begin{tikzcd}
\boxed{FM = HM} \arrow[d, dash] \\
\boxed{Z} \arrow[d, dash] \\
\boxed{T_1}
\end{tikzcd}$$

$(\mathbb{R}, \mathrm{Max})$

$$\begin{tikzcd}
\boxed{Z = HM} \arrow[d, dash] \\
\boxed{FM}
\end{tikzcd}$$

Zariski vs Inverse Zariski

Theorem (Goffer, Greenfeld)
There is a group whose zariski and Inverse zariski topologies are distinct

Theorem (Bardyla, E, Mitchell, Péresse)
If G is a group with $\mathrm{Sym}_\omega(\mathbb{N}) \leq G \leq \mathrm{Sym}(\mathbb{N})$ then the zariski and inverse zariski topologies on G are distinct.

Theorem (Bardyla, E, Mitchell, Péresse)
If G is a group with $F \leq G \leq \mathrm{Homeo}([0,1])$ then the zariski and inverse zariski topologies on G are distinct.

More Zariski Examples

Theorem (de la Nuez Gonzalez, Ghadernezhad, Marimon, Pinsker)
The elementary embedding monoids of the following have zariski strictly coarser than pointwise (agrees with metric ptw)

- rational Urysohn space
- rational Urysohn sphere
- real Urysohn space
- real Urysohn sphere
- rational ultrametric Urysohn space

Theorem (Pinsker, Schindler)
The Zariski topology on the elementary embedding monoid of an ω-categorical structure with no algebraicity is the pointwise topology

When Zariski is not as expected

Theorem (Pinsker, Schindler)
There is an ω-categorical structure A such that the zariski topology on $End(A)$ is not pointwise

Theorem (Bardyla, E, Mitchell, Péresse)
If G is a group with no algebraicity then semigroup zariski is irreducible.

Theorem (E)
If G is a group with $V \leq G \leq Homeo(2^{\omega})$ then group zariski is irreducible

Theorem (de la Nuez Gonzalez, Ghadernezhad, Marimon, Pinsker)
If $G \leq Sym(\mathbb{N})$ is closed with locally finite algebraic closure and non-trivial center then G and $\overline{G}$ have non Hausdorff semigroup Zariski topology.

Theorem (de la Nuez Gonzalez, Ghadernezhad)
If K is a Fraïsse class satisfying certain conditions then the group Zariski topology on $Aut(Flim(K))$ is/isn't a group topology.

e.g $(\mathbb{Q}, <)$ ✓ rational Urysohn space ✗

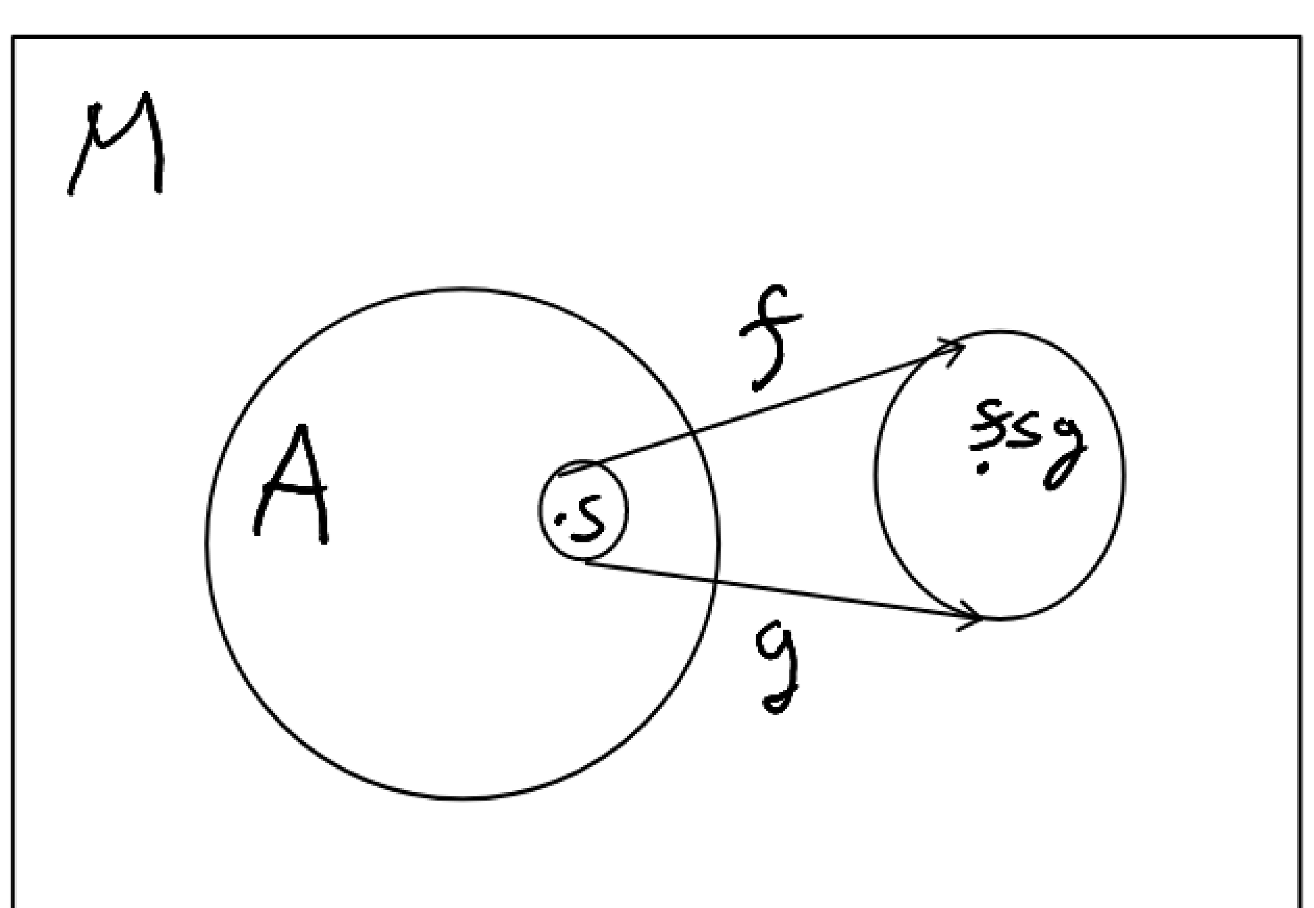
random graph ✗

Standard techniques (upper bounds)

- Ample Generics ⟹ Automatic Continuity (groups)
  (Kechris, Rosendal)

- Standard Borel Spaces (Lusin–Suslin)
  comparable measurable sets ⟹ same measurable sets
  + group ⟹ same topology

- Property X.



⟹ can extend upper bound from A to M

When standard techiques fail

- $\mathrm{End}(\mathbb{Q}, \leq)$ Property $\overline{X}$ (Pinsker, Schindler)

  Unique Polish, non-unique second countable Hausdorff

- $\mathrm{End}(\mathbb{Z}, \leq)$ Property XX (Bardyla, E)

  Unique maximum Polish, infinitely many Polish

- $\mathrm{End}(\mathbb{N}, \leq)$ Property XX (Bardyla, E)

  Unique maximum Polish, infinitely many Polish

(Zariski = pointwise on endomorphisms of tosets)

Counting Polish Topologies

Theorem (E, Jonušas, Mesyan, Mitchell, Morayne, Péresse).
$B_{\mathbb{N}}$ has exactly 0 Polish semigroup topologies.

Theorem (Bardyla, E, Mitchell, Péresse)
$I_{\mathbb{N}}$ has exactly $\aleph_0$ Polish semigroup topologies (classified)

Theorem (Bardyla, E)
$\mathrm{End}(\mathbb{N}, <)$ has exactly $2^{\aleph_0}$ Polish semigroup topologies

Fact (Bardyla, E)
Right zero semigroup on $\mathbb{R}$ has exactly $2^{2^{\aleph_0}}$ Polish semigroup topologies

Questions

- If there is a semigroup with κ Polish topologies is $\kappa \in \{0, 1, \aleph_0, 2^{\aleph_0}, 2^{2^{\aleph_0}}\}$?

- bi-interp result for $I_{\mathbb{N}}$?

- Is there a Polish Group with the small index property but not automatic continuity?

- Does $\mathrm{End}(\mathbb{Z}, \leq)/\mathrm{End}(\mathbb{N}, \leq)$ have automatic continuity?