Infinite Symmetric Groups

Student: Luke Elliott
Supervisor: Dr James D Mitchell

Contents

Chapter 0

Introduction

Abstract

This document is concerned with studying various properties of infinite symmetric groups. These groups are particularly interesting as all groups are isomorphic to permutation groups, and every permutation group is a subgroup of an infinite symmetric group.

0.1 Declaration

I certify that this project report has been written by me, is a record of work carried out by me, and is essentially different from work undertaken for any other purpose or assessment.

0.2 Introduction

Throughout this document we will be working using the ZFC axioms, and in particular, we will make use of the axiom of choice. In this document we aim to prove many results concerning infinite symmetric groups including the following. If Ω is an infinite set then:

1. All elements of $\mathrm{Sym}(\Omega)$ can be written as a commutator of elements of $\mathrm{Sym}(\Omega)$,
2. The group $\mathrm{Sym}(\Omega)$, when viewed as a semigroup, satisfies the semigroup Bergman property,
3. The stong cofinality of $\mathrm{Sym}(\Omega)$ is uncountable,
4. The group $\mathrm{Sym}(\Omega)$ can we written as a product of 10 of its abelian subgroups,
5. There exists a family of $2^{2^{|\Omega|}}$ pairwise non-conjugate maximal subgroups of $\mathrm{Sym}(\Omega)$,
6. We will classify which finite partitions of Ω have maximal subgroups of $\mathrm{Sym}(\Omega)$ as their setwise stabilisers.

This project is organised as follows: We will introduce some notation and definitions that the reader is expected to be reasonably familiar with. We will then spend a chapter proving various results which are not directly related to infinite symmetric groups but will be needed at various points in the document. We will spend the remaining chapters proving various interesting results about infinite symmetric groups.

This project was researched and referenced as follows:

1. The basic definitions and notation section contains only definitions that I was either already familiar with or were directly told to me by my supervisor. These thing are generally considered to be 'common knowledge' in pure mathematics.
2. Chapter 2 is a result of my supervisor directing me towards a proof of the commutator result given at the end of the section. He did this by suggesting theorems for me to prove. However the individual proofs contained in this section were all written by me.
3. In Chapters 1, 3 and 4 most of what is written is based on theorems and proofs taken from papers, with some intermediate proofs constructed by me when I was able to. These proofs have be rewritten in my own words. In some cases these proofs have been altered to be more compatible with this document, to explain further things the original said to be 'clear' or to yield a slightly different result while preserving the fundamental idea of the proof. When a proof is given which is based on the work of another paper this is stated at the start of the proof.

0.3 Basic Definitions and Notation

In this section we will define various terms which are used throughout the document. It is expected that the reader will already be reasonably familiar with most of these terms and therefore we will omit some details and proofs of the validity of these definitions. We will make use of the following notations:

1. If S is a set we use the notation $P(S)$ to denote the power set of S,
2. We use $\mathbb{N}$ to denote the set $\{1, 2, \ldots\}$ and $\mathbb{N}_0$ will be used to denote the set $\{0, 1, 2, \ldots\}$,
3. We use $\subseteq$ to denote a subset and $\subset$ to denote a strict subset,
4. If S is a subset of some set which is clear from context, we use S^c to denote the complement of S,
5. We will use right actions of functions,
6. If A and B are sets then we use A^B to denote the set of functions from B to A,
7. If A and B are sets then $A \sqcup B$ is used to represent the disjoint union of A and B, that is $A \sqcup B := \{(a, 0) : a \in A\} \cup \{(b, 1) : b \in B\}$,
8. If A and B are sets then we use $A \triangle B$ to denote their symmetric difference, that is $A \triangle B := (A \backslash B) \cup (B \backslash A)$,
9. We will use the notations $\langle S \rangle_G$, $\langle S \rangle_S$ and $\langle S \rangle_T$ to denote respectively the group, semigroup and topology generated by the set S,
10. We will use the notation $\leq_G$ to denote a subgroup and the notation $<_G$ to denote a strict subgroup.
11. If a set is ordered in a way which is clear from context we will use $<$ for strictly less than, and $\leq$ for less than or equal to.

Definition 0.3.1 If $f : X \to Y$ is a function then the *domain*, *image*, *fix* and *support* oaf f re defined by:

$$\text{dom}(f) := X, \qquad \text{fix}(f) := \{x \in X : (x)f = x\},$$
$$\text{img}(f) := \{y \in Y : (x)f = y \text{ for some } x \in X\}, \qquad \text{supp}(f) := \{x \in X : (x)f \neq x\}.$$

Definition 0.3.2 Let Ω be an infinite set and let $M \subseteq \Omega$. We call M a *moiety* of Ω if $|M| = |M^c| = |\Omega|$.

Definition 0.3.3 A *partially ordered set* is a pair $(P, \leq)$ where $\leq$ is a subset of $P \times P$ satisfying the below conditions. We use the notation $a \leq b$ to denote $(a, b) \in \leq$ and the notation $a < b$ to denote $a \leq b$ and $a \neq b$.

1. *Reflexive*: For all $p \in P$ we have $p \leq p$,
2. *Anti-Symmetric*: If $a \leq b$ and $b \leq a$ then $a = b$,
3. *Transitive*: If $a \leq b$ and $b \leq c$ then $a \leq c$.

Definition 0.3.4 A *totally ordered set* $(T, \leq)$ is a partially ordered set in which we have, for all $a, b \in T$, that either $a \leq b$ or $b \leq a$.

Definition 0.3.5 A *well ordered set* $(W, \leq)$ is a totally ordered set, in which for all $S \subseteq W$ there exists $m \in S$, such that $m \leq s$, for all $s \in S$.

If R is a function or partial order we will use the notation $R|_S$ to denote R restricted to the elements of S.

Definition 0.3.6 Let $(P, \leq)$ be a partially ordered set and let $C \subseteq P$. We call C a *chain* if $(C, \leq|_C)$ is a totally ordered set.

Definition 0.3.7 Let $(W, \leq)$ be a well ordered set. We call $S \subseteq W$ an *initial segment* of W if $S = \{x \in W : x < M\}$ for some $M \in W$.

Definition 0.3.8 If P_1 and P_2 are partially ordered sets and $\phi : P_1 \to P_2$ is an order preserving bijection, then we call ϕ an *order isomorphism*. In this case we say that P_1 and P_2 are *order isomorphic*.

Definition 0.3.9 A *semigroup* is a pair $(S, *)$ where S is a set, and $* : S \times S \to S$ is a function satisfying the condition below. If $a, b \in S$ then we use the notation $a * b$ (or sometimes just ab) to denote $*(a, b)$.

Associativity: For all $a, b, c \in S$ we have that $(a * b) * c = a * (b * c)$.

Definition 0.3.10 A *group* is a semigroup $(G, *)$ which satisfies the following conditions:

1. *Identity*: There is an element $e \in G$ such that for all $g \in G$ we have $eg = ge = g$,
2. *Inverses*: For all $g \in G$ there exists $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$.

Definition 0.3.11 If G is a group and $f, g \in G$, then we say f and g are *conjugate* if $f = h^{-1}gh$ for some $h \in G$.

Definition 0.3.12 If Ω is a set, then the *symmetric group* $\text{Sym}(\Omega)$ is defined to be the set of bijections $f : \Omega \to \Omega$, under composition of functions.

An infinite symmetric group is simply a symmetric group which acts on an infinite set.

Definition 0.3.13 Let Ω be a set and let S be a set on which $\text{Sym}(\Omega)$ acts.
The *pointwise stabiliser* of S and *setwise stabiliser* of S are defined to be the following:

$$\text{Pstab}(S) = \{f \in \text{Sym}(\Omega) : (x)f = x \text{ for all } x \in S\}, \quad \text{Sstab}(S) = \{f \in \text{Sym}(\Omega) : (x)f \in S \iff x \in S\}.$$

It's not hard to verify that these are groups, and that the pointwise stabiliser is a normal subgroup of the setwise stabiliser.

Definition 0.3.14 Let Ω be a set and let $S \subseteq \Omega$. We introduce the notation $\text{Sym}_\Omega(S) := \text{Pstab}(S^c)$, for the subgroup of $\text{Sym}(\Omega)$ which is naturally isomorphic to $\text{Sym}(S)$.

Definition 0.3.15 Let Ω be a set, let $G \subseteq \text{Sym}(\Omega)$ and let S be a subset of Ω. We say that S is *full* in G or G *acts fully* on S if for all $f \in \text{Sym}(S)$ there exists $f' \in G$ such that $f'|_S = f$. Note that this doesn't necessarily mean that $\text{Sym}_\Omega(S) \subseteq G$.

Definition 0.3.16 Let G be a group which acts on a set S. Then we say G is *transitive* on S if for all $x, y \in S$ there is an $f \in G$ such that $(x)f = y$.

Definition 0.3.17 If Ω is a set, $p \in \Omega$ and $S \subseteq \text{Sym}(\Omega)$ then the *orbit* of p with respect to S is defined by

$$\text{orb}_S(p) := \{x \in \Omega : x = (p)f \text{ for some } f \in \langle S \rangle_G\}.$$

Definition 0.3.18 Let Ω be a set, and let $f \in \text{Sym}(\Omega)$. The term *disjoint cycle shape* of f is used to describe the partition of Ω into equivalence classes under the equivalence relation given by

$$a \sim b \iff a \in \text{orb}_{\{f\}}(b)$$

In particular how many elements of $\Omega / \sim$ there are of each cardinality. If κ is a non-zero cardinal, then the term *κ-cycle* will be used to refer to $f|_P$ where $P \in \Omega / \sim$ and $|P| = \kappa$.
Note that all cycles must have countable domain, as the orbit of a point under a cyclic group is countable. We will also use the notation $(\ldots a_{-1}, a_0, a_1, a_2 \ldots)$ to denote a cycle which maps a_i to a_{i+1} and the notation $(a_0, a_1, a_2 \ldots a_{n-1})$ to denote a cycle which maps a_i to $a_{i+1 \text{ mod n}}$.

Definition 0.3.19 A *metric space* is a pair (X, d) where $d : X \times X \to \mathbb{R}$ is a function satisfying the following conditions:

1. *Non-negativity*: $\text{img}(d) \subseteq [0, \infty)$,
2. *Identity of indiscernibles*: If $x, y \in X$ we have that $d(x, y) = 0$ if and only if $x = y$,
3. *Symmetry*: If $x, y \in X$ we have that $d(x, y) = d(y, x)$,
4. *Triangle inequality*: If $x, y, z \in X$ we have that $d(x, y) \leq d(x, z) + d(y, z)$.

Definition 0.3.20 We say that a metric space is *complete* if every Cauchy sequence is convergent.

Definition 0.3.21 A *topological space* is a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T} \subseteq P(X)$ satisfying the following conditions:

1. We have $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
2. If $S \subseteq \mathcal{T}$ is finite then $\bigcap S \in \mathcal{T}$,
3. If $S \subseteq \mathcal{T}$ then $\bigcup S \in \mathcal{T}$.

Definition 0.3.22 If $(X, \mathcal{T})$ is a topological space then we use the following notations and terminology:

1. If $U \in \mathcal{T}$ we say that U is *open*,
2. If $F^c \in \mathcal{T}$ we say that F is *closed*,
3. If $S \subseteq X$ then we use S° to denote the largest open set contained in S (we call this the *interior* of S),
4. If $S \subseteq X$ then we use $\bar{S}$ to denote the smallest closed set containing S (we call this the *closure* of S),
5. If $(\bar{N})^\circ = \emptyset$ then we say that N is *nowhere-dense*,
6. If M is a countable union of nowhere dense sets then we say that M is *meagre*,
7. If $B \subseteq \mathcal{T}$ satisfies the condition that $\mathcal{T} = \{\bigcup S : S \subseteq B\}$ then we say that B is a *basis* for $\mathcal{T}$,
8. If (X, d) is a metric space, and $\{\{y \in X : d(x, y) < \varepsilon\} : x \in X, \varepsilon > 0\}$ is a basis for $\mathcal{T}$ then we say that $\mathcal{T}$ is *induced* by d.

Definition 0.3.23 If $\{(X_i, \mathcal{T}_i) : i \in I\}$ is a family of topological spaces then we define their *product topology* by

$$\prod_{i \in I}(X_i, \mathcal{T}_i) := (\prod_{i \in I} X_i, \mathcal{T}).$$

Where $\mathcal{T}$ is the topology with basis

$$\left\{ \prod_{i \in I} U_i : U_i \in \mathcal{T}_i \text{ for all } i \text{ and all but finitely many } U_i \text{ are } X_i \right\}.$$

Definition 0.3.24 A subset D of a topological space $(X, \mathcal{T})$ is called *dense* if for all $U \in \mathcal{T} \backslash \{\emptyset\}$ we have $D \cap U \neq \emptyset$. Note this is equivalent to saying $X = \bar{D}$.

Definition 0.3.25 A *topological group* $(X, *, \mathcal{T})$ is a triple where $(X, *)$ is a group and $(X, \mathcal{T})$ is a topological space such that the functions: $* : X \times X \to X$ and $^{-1} : X \to X$ are both continuous (where the topology on $X \times X$ is the product topology and $^{-1}$ is the function which sends an element of X to its group inverse).

Chapter 1

Background Topics

In this chapter we prove various results concerning areas of mathematics that are not directly related to infinite symmetric groups but will be needed in later chapters.

1.1 Topology and Baire Category

In this section we prove some well known results related to baire category and topology which we will need in chapter 2.

Theorem 1.1.1. *If $((X_1, d_1), (X_2, d_2)\ldots)$ are complete metric spaces, then (X_π, d_π) is a complete metric space, where*

$$X_\pi := \prod_{i=1}^{\infty} X_i, \qquad d_\pi((x_{1,1}, x_{1,2}\ldots), (x_{2,1}, x_{2,2}\ldots)) := \sum_{i\in\mathbb{N}} \frac{d_i(x_{1,i}, x_{2,i})}{2^i \times (1 + d_i(x_{1,i}, x_{2,i}))}.$$

Proof. We first show that d_π is a metric. The non-negativity and symmetry conditions follow from the fact that the d_i are non-negative and symmetric for all i. Let

$$x_1 := (x_{1,1}, x_{1,2}\ldots), \qquad x_2 := (x_{2,1}, x_{2,2}\ldots), \qquad x_3 := (x_{3,1}, x_{3,2}\ldots)$$

be arbitrary elements of X_π.
Identity of indiscernibles:

$$d_\pi(x_1, x_1) = \sum_{i\in\mathbb{N}} \frac{d_i(x_{1,i}, x_{1,i})}{2^i \times (1 + d_i(x_{1,i}, x_{1,i}))} = \sum_{i\in\mathbb{N}} \frac{0}{2^i \times (1+0)} = 0,$$

$$d_\pi(x_1, x_2) = 0 \implies \sum_{i\in\mathbb{N}} \frac{d_i(x_{1,i}, x_{2,i})}{2^i \times (1 + d_i(x_{1,i}, x_{2,i}))} = 0 \implies \frac{d_i(x_{1,i}, x_{2,i})}{2^i \times (1 + d_i(x_{1,i}, x_{2,i}))} = 0 \text{ for all } i \in \mathbb{N}$$
$$\implies d_i(x_{1,i}, x_{2,i}) = 0 \text{ for all } i \in \mathbb{N} \implies x_{1,i} = x_{2,i} \text{ for all } i \in \mathbb{N} \implies x_1 = x_2.$$

Triangle inequality: Note that $f : x \to \frac{x}{1+x} = 1 - \frac{1}{1+x}$ is an increasing function on $[0, \infty)$, as it has derivative $\frac{1}{(1+x)^2} > 0$.

$$\begin{aligned}
d_\pi(x_1, x_3) &= \sum_{i\in\mathbb{N}} \frac{d_i(x_{1,i}, x_{3,i})}{2^i \times (1 + d_i(x_{1,i}, x_{3,i}))} \\
&= \sum_{i\in\mathbb{N}} \frac{f(d_i(x_{1,i}, x_{3,i}))}{2^i} \\
&\leq \sum_{i\in\mathbb{N}} \frac{f(d_i(x_{1,i}, x_{2,i}) + d_i(x_{2,i}, x_{3,i}))}{2^i} \\
&= \sum_{i\in\mathbb{N}} \frac{d_i(x_{1,i}, x_{2,i}) + d_i(x_{2,i}, x_{3,i})}{2^i \times (1 + d_i(x_{1,i}, x_{2,i}) + d_i(x_{2,i}, x_{3,i}))} \\
&= \sum_{i\in\mathbb{N}} \frac{d_i(x_{1,i}, x_{2,i})}{2^i \times (1 + d_i(x_{1,i}, x_{2,i}) + d_i(x_{2,i}, x_{3,i}))} + \sum_{i\in\mathbb{N}} \frac{d_i(x_{2,i}, x_{3,i})}{2^i \times (1 + d_i(x_{1,i}, x_{2,i}) + d_i(x_{2,i}, x_{3,i}))} \\
&\leq \sum_{i\in\mathbb{N}} \frac{d_i(x_{1,i}, x_{2,i})}{2^i \times (1 + d_i(x_{1,i}, x_{2,i}))} + \sum_{i\in\mathbb{N}} \frac{d_i(x_{2,i}, x_{3,i})}{2^i \times (1 + d_i(x_{2,i}, x_{3,i}))} \\
&= d_\pi(x_1, x_2) + d_\pi(x_2, x_3).
\end{aligned}$$

We now show that (X_π, d_π) is complete. Let $(x_n)_{n\in\mathbb{N}}$ where $x_n = (x_{n,1}, x_{n,2}\ldots)$ for all $n \in \mathbb{N}$ be a Cauchy sequence. We have that for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$

$$\varepsilon \geq d_\pi(x_n, x_m) = \sum_{i\in\mathbb{N}} \frac{f(d_i(x_{n,i}, x_{m,i}))}{2^i} \geq \frac{f(d_i(x_{n,i}, x_{m,i}))}{2^i} \text{ for all } i \in \mathbb{N}.$$

We have f is a continuous increasing function with $f(0) = 0$, and 2^i is a constant for all i. It follows that for all $i \in \mathbb{N}$ the sequence $(x_{1,i}, x_{2,i}\ldots)$ is Cauchy with respect to d_i. Thus for all $i \in \mathbb{N}$ the sequence $(x_{1,i}, x_{2,i}\ldots)$ is convergent with respect to d_i.
Let $x_l := (x_{l,1}, x_{l,2}\ldots)$ be the sequence of these limits. It suffices to show that $(x_n)_{n\in\mathbb{N}}$ converges to x_l.
Let $\varepsilon > 0$. We have that

$$\sum_{i\in\mathbb{N}} \frac{d_i(x_{n,i}, x_{l,i})}{2^i \times (1 + d_i(x_{n,i}, x_{l,i}))} \leq \sum_{i\in\mathbb{N}} \frac{1}{2^i}$$

for all $n \in \mathbb{N}$. It therefore follows that there exists $k \in \mathbb{N}$, such that for all $n \in \mathbb{N}$ we have

$$\sum_{i=k+1}^{\infty} \frac{d_i(x_{n,i}, x_{l,i})}{2^i \times (1 + d_i(x_{n,i}, x_{l,i}))} \leq \frac{\varepsilon}{2}.$$

As $(x_{1,i}, x_{2,i}\ldots)$ converges to $x_{l,i}$ for all $i \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and $i \in \{1, 2\ldots k\}$ we have

$$\frac{d_i(x_{n,i}, x_{l,i})}{2^i \times (1 + d_i(x_{n,i}, x_{l,i}))} \leq d_i(x_{n,i}, x_{l,i}) \leq \frac{\varepsilon}{2k}.$$

For all $n \geq N$ it follows that

$$d_\pi(x_n, x_l) = \sum_{i\in\mathbb{N}} \frac{d_i(x_{n,i}, x_{l,i})}{2^i \times (1 + d_i(x_{n,i}, x_{l,i}))} = \sum_{i=1}^{k} \frac{d_i(x_{n,i}, x_{l,i})}{2^i \times (1 + d_i(x_{n,i}, x_{l,i}))} + \sum_{i=k+1}^{\infty} \frac{d_i(x_{n,i}, x_{l,i})}{2^i \times (1 + d_i(x_{n,i}, x_{l,i}))}$$

$$\leq \left(\sum_{i=1}^{k} \frac{d_i(x_{n,i}, x_{l,i})}{2^i \times (1 + d_i(x_{n,i}, x_{l,i}))}\right) + \frac{\varepsilon}{2} \leq \left(\sum_{i=1}^{k} \frac{\varepsilon}{2k}\right) + \frac{\varepsilon}{2} = \varepsilon.$$

So $(x_n)_{n\in\mathbb{N}}$ converges to x_l as required. □

Definition 1.1.2 A set S in a topological space is said to be G_δ if S is a countable intersection of open sets.

Definition 1.1.3 A topological space is called *completely metrisable* if it is induced by a complete metric space.

Theorem 1.1.4. *A G_δ subset of a completely metrisable topological space is completely metrisable.*

Proof. The following proof is based on the proof of Theorem 1.2 in [2].
Let $S := \bigcap_{i\in\mathbb{N}} U_i$ for open sets U_i, be a G_δ subset of a completely metrisable topological space $(X, \mathcal{T})$ induced by complete metric space (X, d).
Define the function $\psi : S \to (X \times \mathbb{R} \times \mathbb{R}\ldots)$ by

$$(x)\psi = \left(x, \frac{1}{d(x, U_1^c)}, \frac{1}{d(x, U_2^c)}, \ldots\right)$$

Note that here we do not divide by zero as the U_i^c are closed, so if $d(x, U_i^c) = 0$ then there is a sequence of points in U_i^c converging to x and thus $x \in U_i^c$ a contradiction. By Theorem 1.1.1 we have that $(X \times \mathbb{R} \times \mathbb{R}\ldots, d_\pi)$ is a complete metric space.
<u>Claim:</u> The function $\phi : S \to \mathrm{img}(\psi)$, defined by $(x)\phi = (x)\psi$, is a homeomorphism.

Proof of Claim: By construction ϕ is surjective, and ϕ can be seen to be injective by considering the first coordinate of the image. We need to show ϕ is continuous. Let $\varepsilon > 0$ and $x \in S$. Let $k \in \mathbb{N}$ be such that $\sum_{i=k+1}^{\infty} \frac{1}{2^{i+1}} \leq \frac{\varepsilon}{3}$ and let

$$\delta := \min\left\{\frac{\varepsilon}{3}, \frac{\min\{d(x, U_i^c) : i \in \{1, 2\ldots k\}\}}{2}, \frac{\varepsilon \min\{d(x, U_i^c) : i \in \{1, 2\ldots k\}\}^2}{6k}\right\}$$

Let $y \in S$ be such that $d(x, y) < \delta$. For ϕ to be continuous it suffices to show that $d_\pi((x)\phi, (y)\phi) \leq \varepsilon$.

$$d_\pi((x)\phi, (y)\phi) = \frac{d(x, y)}{1 + d(x, y)} + \sum_{i=1}^{\infty} \frac{1}{2^{i+1}} \left(\frac{\left|\frac{1}{d(x,U_i^c)} - \frac{1}{d(y,U_i^c)}\right|}{1 + \left|\frac{1}{d(x,U_i^c)} - \frac{1}{d(y,U_i^c)}\right|}\right) \leq d(x, y) + \frac{\varepsilon}{3} + \sum_{i=1}^{k} \frac{1}{2^{i+1}} \left(\frac{\left|\frac{d(y,U_i^c) - d(x,U_i^c)}{d(x,U_i^c)d(y,U_i^c)}\right|}{1 + \left|\frac{1}{d(x,U_i^c)} - \frac{1}{d(y,U_i^c)}\right|}\right)$$

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{i=1}^{k} \left| \frac{d(y,U_i^c) - d(x,U_i^c)}{d(x,U_i^c)d(y,U_i^c)} \right| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{i=1}^{k} \frac{d(x,y)}{d(x,U_i^c)d(y,U_i^c)} \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{i=1}^{k} \frac{d(x,y)}{d(x,U_i^c)(d(x,U_i^c) - \delta)}$$
$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{i=1}^{k} \frac{d(x,y)}{d(x,U_i^c)\frac{d(x,U_i^c)}{2}} \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{i=1}^{k} \frac{2\delta}{d(x,U_i^c)d(x,U_i^c)} \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{i=1}^{k} \frac{2\varepsilon \min\{d(x,U_i^c) : i \in \{1,2\dots k\}\}^2}{6kd(x,U_i^c)^2}$$
$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{i=1}^{k} \frac{\varepsilon}{3k} \leq \varepsilon.$$

Finally we need to show ϕ^{-1} is continuous. Let $\varepsilon > 0$ and $x \in \text{img}(\phi)$, let $\delta := \min\{\frac{\varepsilon}{2}, \frac{1}{2}\}$ and let $y \in \text{img}(\phi)$.

$$d_\pi(x,y) < \delta \implies \frac{d((x)\phi^{-1},(y)\phi^{-1})}{1 + d((x)\phi^{-1},(y)\phi^{-1})} < \delta \implies d((x)\phi^{-1},(y)\phi^{-1}) < \delta(1 + d((x)\phi^{-1},(y)\phi^{-1}))$$
$$\implies d((x)\phi^{-1},(y)\phi^{-1})(1-\delta) < \delta \implies d((x)\phi^{-1},(y)\phi^{-1}) < \frac{\delta}{1-\delta} \leq 2\delta \leq \varepsilon. \quad \square$$

We have that S is homeomorphic to img(ϕ) which is contained in a complete metric space. So to show S is completely metrisable it suffices to show that img(ϕ) is closed and therefore complete.
Let $((x_n)\phi)_{n\in\mathbb{N}}$ be a sequence in img(ϕ) which converges to $(x, r_1, r_2 \dots)$ in $(X \times \mathbb{R} \times \mathbb{R} \dots)$. For all $i \in \mathbb{N}$ we now have

$$r_i = \lim_{n\to\infty} \frac{1}{d(x_n,U_i^c)} = \frac{1}{d(\lim_{n\to\infty} x_n, U_i^c)} = \frac{1}{d(x,U_i^c)}$$
$$\implies d(x,U_i^c) \neq 0 \implies x \notin U_i^c \implies x \in U_i.$$

It follows that $x \in \bigcap_{i\in\mathbb{N}} U_i = S$. As $x \in S$ and $\frac{1}{d(x,U_i^c)} = r_i$ for all $i \in \mathbb{N}$, we have that $(x, r_1, r_2, \dots) = (x)\phi \in \text{img}(\phi)$ as required. $\square$

In fact it is also true that if a subspace of a completely metrizable space is completely metrizable, then this subspace is G_δ. This is proven in [2]. However this fact is not required in this document so the proof is omitted.

Theorem 1.1.5. *If N is nowhere dense in a topological space $(X, \mathcal{T})$, then N^c is dense.*

Proof. Suppose for a contradiction that U is a non-empty open set satisfying $U \cap N^c = \emptyset$.

$$U \cap N^c = \emptyset \implies U \backslash N = \emptyset \implies U \subseteq N \implies U \subseteq \overline{N} \implies U \subseteq (\overline{N})^\circ \implies (\overline{N})^\circ \neq \emptyset.$$

The penultimate implication follows because U is open. We therefore have that $(\overline{N})^\circ \neq \emptyset$, which is a contradiction as N is nowhere dense. $\square$

Theorem 1.1.6. *Let $(X, \mathcal{T})$ be a topological space induced by a metric d. A set N is nowhere dense if and only if for all $x_1 \in X$ and $r_1 > 0$, there exists an $x_2 \in X$ and an $r_2 > 0$ such that $B(x_2, r_2) \subseteq B(x_1, r_1)\backslash N$.*

Proof. ($\Rightarrow$) Let N be nowhere dense. Suppose for a contradiction that there exists $x_1 \in X$ and $r_1 > 0$ such that there are no $x_2 \in X$ and $r_2 > 0$ satisfying $B(x_2, r_2) \subseteq B(x_1, r_1)\backslash N$.
Let $x \in B(x_1, r_1)$. As $B(x_1, r_1)$ is open there exists an $r > 0$ such that $B(x, r) \subseteq B(x_1, r_1)$. By assertion $B(x, r) \nsubseteq B(x_1, r_1)\backslash N$, so there exists $y \in N \cap B(x, r)$. As r can be made arbitrarily small, we therefore have that $x \in \bar{N}$. As x was arbitrary it follows that $B(x_1, r_1) \subseteq \bar{N}$, and therefore $B(x_1, r_1) \subseteq (\overline{N})^\circ$. So $x_1 \in (\overline{N})^\circ$, this is a contradiction as N is nowhere dense.
($\Leftarrow$) Suppose for a contradiction that $(\overline{N})^\circ \neq \emptyset$. Let $x \in (\overline{N})^\circ$. As $(\overline{N})^\circ$ is open there exists $r > 0$ such that $B(x, r) \subseteq (\overline{N})^\circ$. By assertion there exists $x_2 \in X$ and $r_2 > 0$ such that $B(x_2, r_2) \subseteq B(x, r)\backslash N \subseteq (\overline{N})^\circ\backslash N \subseteq \overline{N}\backslash N$. We therefore have $x_2 \in B(x_2, r_2) \subseteq \overline{N}\backslash N$ and in particular $x \in \bar{N}$.
But $B(x_2, r_2)$ is open and contains x_2 and $B(x_2, r_2) \subseteq B(x, r)\backslash N$ so $B(x_2, r_2) \cap N = \emptyset$. This contradicts the fact that $x_2 \in \bar{N}$. $\square$

Definition 1.1.7 A *Baire Space* is a topological space in which any countable collection of dense open sets $(U_n)_{n\in\mathbb{N}}$ has dense intersection.

Theorem 1.1.8 (Baire Category Theorem)**.** *Every completely metrisable topological space is a Baire Space. In addition, a non-empty completely metrisable space is not meagre.*

Proof. The following proof is based on the proof of the Baire Category Theorem found in [1].
Let $(X, \mathcal{T})$ be a completely metrisable topological space induced by the complete metric d. Let $(U_n)_{n\in\mathbb{N}}$ be a countable collection of dense open sets.

We want to show that $I = \bigcap_{i\in\mathbb{N}} U_i$ is dense in X. Let $x \in X$. As x is arbitrary it suffices to show that $x \in \bar{I}$.
Let U_x be an open set containing x. It suffices to show that $I \cap U_x \neq \emptyset$. As $\mathcal{T}$ is induced by d, there exists an $r_0 > 0$ such that $B(x, r_0) \subseteq U_x$.
Let B_0 and V_0 be $B(x, r_0)$. Let $V_n := U_n \cap B_{n-1}$ for all $n \in \mathbb{N}$ and $B_n := B(x_n, r_n)$ for all $n \in \mathbb{N}$ be such that $B_n \subseteq B(x_n, 2r_n) \subseteq V_n$ and $0 < r_n < \frac{r_{n-1}}{2}$.
Note that this construction is possible as each V_i is constructed by intersecting an open set with an open set, so each V_i is open.
In addition V_i is non-empty as the intersection of a dense set and a non-empty open set.
We will now show that $(x_n)_n$ is a Cauchy sequence. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ large enough that $\frac{r_0}{2^N} < \frac{\varepsilon}{2}$. For $n \geq N$ we have $x_n \in B_n \subseteq V_n \subseteq B_{n-1} \ldots\ldots \subseteq B_N = B(x_N, r_N) \subseteq B(x_N, \frac{r_0}{2^N})$. This final inclusion follows as each B has radius less than half the size of the previous one. So we have that $d(x_n, x_N) < \frac{r_0}{2^N} < \frac{\varepsilon}{2}$.
So for $n, m \geq N$ we have
$$d(x_n, x_m) \leq d(x_n, x_N) + d(x_m, x_N) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
We therefore have that $(x_n)_n$ is Cauchy.
As (X, d) is complete we have that $(x_n)_n$ is convergent. Let $y := \lim_{n\to\infty} x_n$. It suffices to show that $y \in U_x$ and $y \in I$. For $k \in \mathbb{N}$, the sequence $(x_{k+1}, x_{k+2}, \ldots)$ converges to y. For all $i > k$ we have
$$x_i \in B_i \subseteq B_{i-1} \ldots \subseteq B_{k+1} \subseteq \overline{B_{k+1}} = \{z \in X : d(x_{k+1}, z) \leq r_{k+1}\} \subseteq \{z \in X : d(x, z) < 2r_{k+1}\} \subseteq V_{k+1} \subseteq B_k.$$
As for all $i > k$ we have $x_i \in \overline{B_{k+1}}$ a closed set, we have that $y \in \overline{B_{k+1}} \subseteq B_k \subseteq V_k \subseteq U_k$. We therefore have that $y \in \bigcap_{i\in\mathbb{N}} U_i = I$, and $y \in B_1 \subseteq B_0 \subseteq U_x$ as required. □
Suppose for a contradiction that $(X, \mathcal{T})$ is a non-empty completely metrisable space such that $X = \bigcup_{i\in\mathbb{N}} N_i (= \bigcup_{i\in\mathbb{N}} \overline{N_i})$ where the N_i are nowhere dense for all $i \in \mathbb{N}$. As $(X, \mathcal{T})$ is completely metrisable we have that $(X, \mathcal{T})$ is a Baire space.
Consider the sets $(\overline{N_i}^c)_{i\in\mathbb{N}}$. As the complements of closed nowhere dense sets, these sets are open and dense. We therefore have the following
$$\bigcap_{i\in\mathbb{N}} \overline{N_i}^c \text{ is dense} \implies \left(\bigcup_{i\in\mathbb{N}} \overline{N_i}\right)^c \text{ is dense} \implies (X)^c \text{ is dense} \implies \emptyset \text{ is dense.}$$
As $(X, \mathcal{T})$ is a non-empty topological space, X is a non-empty open set. But $X \cap \emptyset = \emptyset$, this contradicts the fact that $\emptyset$ is dense. □

Theorem 1.1.9. *If $(X, *, \mathcal{T})$ is a topological group, then we have that for all $x \in X$ the functions given by:*
$$(y)\phi_{x_r} = yx, \qquad (y)\phi_{x_l} = xy$$
are homeomorphisms.

Proof. Let $x \in X$, as ϕ_{x_r} has the inverse $\phi_{{x^{-1}}_r}$ and ϕ_{x_l} has inverse $\phi_{{x^{-1}}_l}$ we have that these are bijections. In addition we have by symmetry that if ϕ_{x_r} is continuous then $\phi_{x_r}^{-1}, \phi_{x_l}$ and $\phi_{x_l}^{-1}$ are continuous. It therefore suffices to show that ϕ_{x_r} is continuous.
Let U be open in $\mathcal{T}$. We will show that $(U)\phi_{x_r}^{-1}$ is open.
We have that $(U)*^{-1}$ is open. By definition of the product topology, this means that there exists a collection of open sets B such that $\cup B = (U)*^{-1}$ and for all $b \in B$ we have $b = U_{b,1} \times U_{b,2}$ for open sets $U_{b,1}$ and $U_{b,2}$. As $U_{b,1}$ is open for all $b \in B$, it suffices to show that $(U)\phi_{x_r}^{-1} = \cup\{U_{b,1} : x \in U_{b,2}\}$.
$$\begin{aligned} y \in (U)\phi_{x_r}^{-1} &\iff yx \in U \iff (y, x) \in (U)*^{-1} \iff (y, x) \in \cup B \\ &\iff \text{there exists } b \in B \text{ such that } (y, x) \in b \\ &\iff \text{there exists } b \in B \text{ such that } y \in U_{b,1} \text{ and } x \in U_{b,2} \\ &\iff y \in \cup\{U_{b,1} : x \in U_{b,2}\}. \end{aligned}$$
We therefore have that $(U)\phi_{x_r}^{-1} = \cup\{U_{b,1} : x \in U_{b,2}\}$ and is thus open as required. □

1.2 Ordinals

Definition 1.2.1 An *ordinal* is a set α satisfying the following conditions:

1. Transitivity: If $x \in \alpha$ and $y \in x$ then $y \in \alpha$,
2. Well ordered: The pair $(\alpha, \{(a, b) \in \alpha \times \alpha : \text{we have precisely one of } a \in b \text{ or } a = b)\})$ is a well ordered set.

It is not hard to see that if α is an ordinal then $\alpha^+ := \alpha \cup \{\alpha\}$ is also an ordinal. This is in fact the smallest ordinal containing alpha.

Definition 1.2.2 If α is an ordinal then $\alpha + 0 := \alpha$ and $\alpha + k := (\alpha + (k-1))^+$ for all $k \in \mathbb{N}$. If $\alpha \neq \beta^+$ for any ordinal β then we say that α is a *limit ordinal.*

Theorem 1.2.3. *If α is an ordinal then $\alpha = \beta + k$ for some limit ordinal β and $k \in \mathbb{N}$.*

Proof. Suppose that $\alpha \neq \beta + k$ for any limit ordinal β and $k \in \mathbb{N}$. It follows that α is not a limit ordinal so $\alpha = \alpha_1 + 1$ for some ordinal α_1. It follows that α_1 is not a limit ordinal so there exists an ordinal α_2 such that $\alpha_1 = \alpha_2 + 1$ and thus $\alpha = \alpha_2 + 2$. By repeating this we can construct a decreasing sequence of ordinals $(\alpha_n)_n \subseteq \alpha$. This sequence has no minimum element which contradicts the fact that α is well ordered. □

Theorem 1.2.4. *Every element of an ordinal is an ordinal.*

Proof. This proof is based on the proof of Theorem 2.6 in [7].
Let α be an ordinal and let $x \in \alpha$.
Transitivity: If $z \in y \in x$ then we have that $y \in \alpha$ and $z \in \alpha$ by the transitivity of α. Also $z < y < x$ so by the transitivity of α as a well ordered set we have $z < x$ and thus $z \in x$.
Well ordered: By the transitivity of α we have that $x \subseteq \alpha$ and therefore is well ordered by the required ordering. □

In the following theorem we use the word *class* to refer to all the sets with a specific property.

Theorem 1.2.5. *If C is a non-empty class of ordinals then it has a least element, that is an element contained in all other elements.*

Proof. This proof is based on the proof if Theorem 2.11 in [7].
We will show that $I := \bigcap C$ is the desired element.
<u>Claim:</u> If $\alpha \in C$ and $\alpha \neq I$ then $\min(\alpha \backslash I) = I$.
Proof of Claim: Let $\alpha \in C \backslash I$. Let $x \in \min(\alpha \backslash I)$. We have that $x < \min(\alpha \backslash I)$ and $x \in \alpha$ by transitivity so $x \in I$. Conversely let $x \in I$. We want to show that $x \in \min(\alpha \backslash I)$. As $x \in I$ we have that $x \neq \min(\alpha \backslash I)$. Suppose for a contradiction that $x > \min(\alpha \backslash I)$. Then $\min(\alpha \backslash I) \in x$ and $x \in I$, so by transitivity of ordinals $\min(\alpha \backslash I)$ is an element of every element of C, and is thus an element of I, a contradiction. It follows that $x < \min(\alpha \backslash I)$ and thus $x \in \min(\alpha \backslash I)$ as required. □
It follows from the Claim that if $I \in C$ then I is contained in all other elements as required. Suppose for a contradiction that $I \notin C$. By the Claim we have that I is an element of every element of C and thus $I \in \cap C = I$. As I is an element of the elements of C which are ordinals, we have that I is an ordinal. Therefore I is well ordered with $I < I$, a contradiction. □

Corollary 1.2.6. *If α and β are ordinals then we have one of $\alpha \in \beta$, $\beta \in \alpha$ or $\alpha = \beta$.*

Corollary 1.2.7. *All transitive sets of ordinals are ordinals.*

Theorem 1.2.8. *If I_1 and I_2 are initial segments of a well ordered set and there is an order isomorphism $\phi : I_1 \to I_2$, then ϕ is the identity map and thus $I_1 = I_2$.*

Proof. Let $\phi : I_1 \to I_2$ be an order preserving bijection. Suppose for a contradiction that an element of I_1 is not fixed by ϕ. Let m be the minimum such element. It follows that $(m)\phi \neq m$. If $(m)\phi < m$ then $((m)\phi)\phi = (m)\phi$, contradicting the injectivity of ϕ. It follows that $m < (m)\phi \in I_2$. Thus we have $m \in I_2 = \text{img}(\phi)$. Similarly, if $(m)\phi^{-1} < m$ then $((m)\phi^{-1})\phi^{-1} = (m)\phi^{-1}$, contradicting the injectivity of ϕ^{-1}. It follows that we have both $(m)\phi > m$ and $(m)\phi^{-1} > m$. As ϕ is order preserving we conclude that $(m)\phi > m$ and $m = ((m)\phi^{-1})\phi > (m)\phi$. This is a contradiction. □

Corollary 1.2.9. *If α and β are ordinals and there is a bijection $\phi : \alpha \to \beta$ which preserves order. Then ϕ is the identity map and thus $\alpha = \beta$.*

Proof. Observe that α and β are both initial segments of $\max(\alpha, \beta)^+$. □

Definition 1.2.10 A *cardinal* is defined to be an ordinal which in not in bijective correspondence with any lesser ordinal.

It will be shown in the next section that all sets are in bijective correspondence with an ordinal so this is a reasonable way to view cardinality of sets in general. It will be useful to notice that all infinite cardinals are limit ordinals. We will use the symbol $\aleph_0$ to denote the smallest infinite cardinal. This set is often used as a definition for the naturals numbers (with 0) and its elements will be viewed as the natural numbers in this document.

1.3 Infinite sets

In this section we will prove various theorems concerning infinite sets which will be used throughout this document.

Theorem 1.3.1 (Zorn's Lemma)**.** *If $(P, \leq)$ is a partially ordered set such that every chain has an upper bound, then P has a maximal element M, that is an element such that there is no $x \in P$ with $x > M$.*

Proof. The following proof is based of the proofs of Lemma 3.3 and Theorem 4.2 in [11].
Suppose for a contradiction there is no maximal element. By the axiom of choice there is a choice function c for $\mathcal{P}(P)\backslash\{\emptyset\}$. For $a \in P$ let $G_a := \{x \in P : x > a\}$. For $C \subseteq P$, a chain, let $G_C := \{x \in P : x \text{ is an upper bound for } C\}$. As P has no maximal element and every chain has an upper bound we have that these sets are non-empty.
We call a chain C a c-chain if it satisfies the following:

1. C is well ordered by $\leq$,
2. If $C_M \subset C$ is an initial segment of C with maximal element M then $\min(C\backslash C_M) = c(G_M)$,
3. If $C_u \subset C$ is an initial segment of C with no maximal element then $\min(C\backslash C_u) = c(G_{C_u})$.

Claim 1: If $C_1 \neq C_2$ are c-chains, and all initial segments of C_1 are initial segments of C_2, then C_1 is an initial segment of C_2.
Proof of Claim: If C_1 has a maximal element m then it follows from the definition of a c-chain that

$$m = \min(C_1\backslash\{x \in C_1 : x < m\}) = \min(C_2\backslash\{x \in C_1 : x < m\}).$$

It follows that $C_1 = \{x \in C_2 : x < \min(C_2\backslash C_1)\}$ is an initial segment of C_2. If C_1 has no maximal element then it follows that $C_1 \subset C_2$, as all elements of C_1 are contained in an initial segment bounded above by a greater element of C_1. It again follows that $C_1 = \{x \in C_2 : x < \min(C_2\backslash C_1)\}$ is an initial segment of C_2. □
Claim 2: If $C_1 \neq C_2$ are c-chains then one is an initial segment of the other.
Proof of Claim: Suppose that C_2 is not an initial segment of C_1. By Claim 1 it suffices to show that all initial segments of C_1 are initial segments of C_2. Suppose for a contradiction that this is not the case. The initial segments of C_1 are naturally well ordered by containment. There is therefore least initial segment I of C_1 which is not an initial segment of C_2. As I is an initial segment of a c-chain we have that I is a c-chain. As I is the minimal initial segment of C_1 which is not an initial segment of C_2 we have that all initial segments of I are initial segments of C_2. It therefore follows by Claim 1 that I is an initial segment of C_2 or $C_2 = I$. Both of these cases are contradictions as I is not an initial segment of C_2 by definition, and C_2 is not an initial segment of C_1. □
Let S be the union of all c-chains of P. Note that for all elements $x \in S$ there exists a c-chain C_x such that $x = \max(C_x)$. This follows as x is contained in a c-chain, and if we restrict that c-chain to the elements less than or equal to x, then we also have a c-chain. We will now show that S is a c-chain.

1. Let $s, t \in S$ and let C_s and C_t be c-chains with s and t maximal respectively. By Claim 2, we have that either $C_s \subseteq C_t$ or $C_t \subseteq C_s$, so either $s \leq t$ or $t \leq s$ and therefore S is totally ordered. Let $A \subseteq S$ be non-empty and let $t \in A$. Consider the set $B := \{x \in A : x < t\}$. If B is empty then t is minimal in A, otherwise if B has a least element then A has a least element and so S is well ordered. All $x \in B$ are contained in a c-chain C_x with x as maximal element. If $C_x \supset C_t$ it follows that $t \leq x$, a contradiction, so we have $C_x \subseteq C_t$ and thus $B \subseteq C_t$ so B has a minimal element and S is well ordered.
2. Let $S_M \subset S$ be an initial segment with maximal element M. Let $s \in S\backslash S_M$. It follows that there is a c-chain $C_s \supseteq S_M$ with maximal element s, and thus $c(G_M) = \min\{y \in C_s : y \notin \{x \in C_s : x \leq M\}\} \in S$. As $c(G_M) \in C_s$ it follows that $c(G_M) \leq s$ but s was arbitrary so we have that $\min\{x \in S : x \notin S_M\} = c(G_M)$ as required.
3. Let $S_u \subset S$ be an initial segment with no maximal element. Let $s \in S\backslash S_u$. Let C_s be a c-chain with s as its maximal element. If $x \in S_u$ it follows there is a c-chain with x maximal, which is an initial segment of C_s and thus $x \in C_s$. So we have that $S_u \subset C_s \subseteq S$. As C_s is a c-chain it follows that $c(G_{S_u}) = \min\{y \in C_s : y \notin S_u\} \in S$ and, as s was arbitrary and $s \geq c(G_{S_u})$, we have that $\min\{x \in S : x \notin C_u\} = c(G_{S_u})$.

Now we have that S is a c-chain and there is no greater c-chain than S by definition. However if S has a maximal element M then $S \cup \{c(G_M)\}$ is a strictly greater c-chain and if S has no maximal element we have that $S \cup \{c(G_S)\}$ is a strictly greater c-chain so we have reached a contradiction. □

Theorem 1.3.2. *Every set A is well orderable.*

Proof. The following proof is based of the proof of the same Theorem given in [10].
Let $A_o := \{(S, \leq) : S \subseteq A, \leq \text{ is a well ordering of S}\}$. Let A_o be partially ordered by

$$(S_1, \leq_1) \leq (S_2, \leq_2) \iff (S_1 \subseteq S_2 \text{ and } \leq_2|_{S_1} = \leq_1 \text{ and for all } x_1 \in S_1, x_2 \in S_2\backslash S_1 \text{ we have } x_1 \leq_2 x_2).$$

Let $C \subseteq A_o$ be a chain. We claim that the tuple

$$(S_C, \leq_C) := (\{a \in A : a \in S \text{ for some } (S, \leq) \in C\}, \{(a_1, a_2) \in A \times A : a_1 \leq a_2 \text{ for some } (S, \leq) \in C\})$$

is an upper bound for C. Given any two elements $x, y \in S_C$ there is a well ordered set containing both x and y (each is contained in an element of C so take the larger of these elements). So we have one of $x \leq_C y$ or $x \geq_C y$. Finally, if $D \subseteq S_C$ is non-empty then there exists $(S, \leq) \in C$ such that $S \cap D \neq \emptyset$. It follows that $\min(S \cap D)$ is minimal in D, as by the definition of the ordering on A_o we have all elements of $D \backslash S$ are greater than it. So we have that $(S_C, \leq_C) \in A_o$. By construction we also have that it is greater then all elements of C.
By Zorn's Lemma, A_o has a maximal element $(X, \leq_x)$. If $X \neq A$ then there is an element $a \in A \backslash X$. Therefore we have that $(X \cup \{a\}, \leq_x \cup \{(x, a) : x \in X \cup \{a\}\} > (X, \leq_x)$, a contradiction. So we must have that $A = X$, and is well ordered by $\leq_x$. □

Theorem 1.3.3. *Every well ordered set $(S, \leq)$ is order isomorphic to an ordinal.*

Proof. For $x \in S$, let $I_x := \{y \in S : y < x\}$. Note that the initial segments of S are well ordered by $I_x \leq I_y \iff x \leq y$. Suppose for a contradiction that there are initial segments of S which are not order isomorphic to an ordinal. Let I_m be the minimal such initial segment. As ordinals are only order isomorphic if they are equal, it follows that for all $I_x < I_m$ there is a unique ordinal α_x order isomorphic to I_x. By the axiom of replacement it follows that $\alpha_m := \{\alpha_x : x < m\}$ is a well-defined set. Let $\phi : I_m \to \alpha_m$ be defined by $\phi(x) = \alpha_x$. We have that ϕ is injective as different initial segments can't be order isomorphic and it is surjective by the definition of α_m. It must also preserve order as the initial segments are initial segments of each other and thus the order isomorphisms are extensions of each other. Let $\alpha \in \alpha_x \in \alpha_m$, let $\psi : I_x \to \alpha_x$ be an order isomorphism. It follows that $\psi|_{I_{(\alpha)\psi^{-1}}}$ is an order isomorphism between α and an initial segment of S, so $\alpha \in \alpha_m$. We now have that α_m is a transitive set of ordinals and is thus an ordinal. In addition ϕ is an order isomorphism from I_m to α_m so we have contradicted the definition of I_m.
We now have that all initial segments of S are order isomorphic to an ordinal. Therefore we can apply the same reasoning we did to I_m to conclude that S is order isomorphic to an ordinal as required. □

By the previous two theorems we now have that for every set, there exists an ordinal in bijective correspandance with it. Thus we may now safely assign cardinals to any set. If S is a set we will use the notation $|S|$ to denote its *cardinality* (the unique cardinal in bijective correspondence with it). For cardinals α and β, we now define the standard cardinal operations.

$$\alpha + \beta := |\alpha \sqcup \beta|, \quad \alpha\beta := |\alpha \times \beta|, \quad \alpha^\beta := |\alpha^\beta|.$$

Theorem 1.3.4. *If Ω is an infinite set, then $|\Omega| = 2|\Omega|$.*

Proof. Without loss of generality we may assume that Ω is a cardinal. Let $\phi : \Omega \to \Omega \sqcup \Omega$ be defined by

$$(x)\phi = \left\{ \begin{array}{ll} (\alpha + (k/2), 0) & \text{if } x = \alpha + k \text{ for a limit ordinal } \alpha \text{ and } k \in \aleph_0 \text{ even} \\ (\alpha + ((k-1)/2), 1) & \text{if } x = \alpha + k \text{ for a limit ordinal } \alpha \text{ and } k \in \aleph_0 \text{ odd} \end{array} \right\}.$$

We have that ϕ is a bijection and therefore $|\Omega| = |\Omega \sqcup \Omega| = 2|\Omega|$. □

Corollary 1.3.5. *If Ω is an infinite set, then it has moiety subsets.*

Theorem 1.3.6. *If Ω is an infinite set, then $|\Omega| = |\Omega|^2$.*

Proof. Without loss of generality we can assume Ω is a cardinal. Let $\Omega_o := \{(S, \phi) : S \subseteq \Omega \text{ is infinite}, \phi : S \to S \times S \text{ is a bijection}\}$. Let Ω_o be partially ordered by: $(S_1, \phi_1) \leq (S_2, \phi_2) \iff S_1 \subseteq S_2$ and $\phi_2|_{S_1} = \phi_1$. Note that Ω_o is non-empty as there is a bijection from $\aleph_0$ to $\aleph_0 \times \aleph_0$.
Let $((S_i, \phi_i))_{i \in I}$ be a chain, where I is an index set. Let $\phi_U : \cup_{i \in I} S_i \to \cup_{i \in I} S_i \times \cup_{i \in I} S_i$ be defined by

$$(x)\phi_U = (x)\phi_i \text{ for } x \in S_i.$$

This is a well-defined bijection as the bijections agree whenever they are defined. Therefore $(\bigcup_{i \in I} S_i, \phi_U)$ is an upper bound for our chain.
By Zorn's Lemma, Ω_o has a maximal element (X, ϕ_x). If $|X| = |\Omega|$ then we are done as there are bijections from Ω to X and $\Omega \times \Omega$ to $X \times X$. Suppose for a contradiction that $|X| < |\Omega|$. It follows that $|X| < |X^c|$ and so there exists $X' \subseteq X^c$ such that

$$|X| = |X'| = |X' \times X'| = 3|X' \times X'| = |X' \times X'| + |X' \times X| + |X \times X'| = |(X' \times X') \cup (X' \times X) \cup (X \times X')|.$$

Let $\phi'_x : X' \to (X' \times X') \cup (X' \times X) \cup (X \times X')$ be a bijection. By adjoining the functions ϕ_x and ϕ'_x we can construct a bijection $\phi''_x : X \cup X' \to (X \cup X') \times (X \cup X')$. We therefore have that $(X, \phi_x) < (X \cup X', \phi''_x)$ a contradiction. □

Theorem 1.3.7. *An infinite symmetric group* $\mathrm{Sym}(\Omega)$ *has cardinality* $2^{|\Omega|}$.

Proof. Let S be a subset of Ω. If S is finite, then there exists a bijection $\phi : S \to \{0, 1, 2, \ldots (n-1)\}$ for some $n \in \mathbb{N}_0$. We can construct a bijection $f_S : \Omega \to \Omega$ as follows:

$$(x)f_S = \left\{ \begin{array}{ll} ((x)\phi + 1 \text{ mod n})\phi^{-1} & x \in S \\ x & \text{otherwise} \end{array} \right\}.$$

If S is infinite then we can construct a partition $\{M_1, M_2\}$ of S such that M_1, M_2 are moieties of S. There exists a bijection $\phi : M_1 \to M_2$ so we can construct a bijection $f_S : \Omega \to \Omega$ as follows:

$$(x)f_S = \left\{ \begin{array}{ll} (x)\phi & x \in M_1 \\ (x)\phi^{-1} & x \in M_2 \\ x & \text{otherwise} \end{array} \right\}.$$

For $|S| \geq 2$ we have $\mathrm{supp}(f_S) = S$ and therefore the f_S are all distinct.

$$2^{|\Omega|} = 2^{|\Omega|} - |\Omega| = |P(\Omega) \backslash \{\{\emptyset\} \cup \{\{x\} : x \in \Omega\}\}| \leq |\mathrm{Sym}(\Omega)| \leq |\Omega^{\Omega}| \leq |(2^{|\Omega|})^{|\Omega|}| = |2^{|\Omega \times \Omega|}| = 2^{|\Omega|}.$$

Therefore we have that $|\mathrm{Sym}(\Omega)| = 2^{|\Omega|}$. □

1.4 Ultrafilters

In this section we will prove various facts about ultrafilters which will be needed in the first section of chapter 4.

Definition 1.4.1 Given a set Ω, a *filter* on Ω is defined to be a collection of subsets F of Ω satisfying:

1. We have $\Omega \in F$,
2. For all $A, B \in F$ we have $A \cap B \in F$,
3. If $A \subseteq B \subseteq \Omega$ and $A \in F$ then $B \in F$.

Example 1.4.2 If Ω is a set then the following are filters on Ω.

1. The set $\{\Omega\}$.
2. The set $P(\Omega)$.
3. The set $\{X \subseteq \Omega : |\Omega \backslash X| < \kappa\}$, where κ is any infinite cardinal.
4. The set $\{X \subseteq \Omega : S \subseteq X\}$, where S is any subset of Ω.

Definition 1.4.3 Given a set Ω, an *ultrafilter* on Ω is defined to be a filter $\mathcal{U}$ on Ω satisfying:

1. The empty set is not an element of $\mathcal{U}$,
2. There is no filter $\mathcal{U}'$ such that $\mathcal{U} \subset \mathcal{U}' \subset P(\Omega)$.

Example 1.4.4 If Ω is a set and $x \in \Omega$ then $\{X \subseteq \Omega : x \in X\}$ is an ultrafilter on Ω.

Theorem 1.4.5. *Let* Ω *be a set, let* $S \subseteq \Omega$ *and let* $\mathcal{U}$ *be an ultrafilter on* Ω*. Then precisely one of* S *and* S^c *is in* $\mathcal{U}$.

Proof. If we had that both S and S^c were in $\mathcal{U}$ then $S \cap S^c = \emptyset$ would also be in $\mathcal{U}$, a contradiction. Suppose for a contradiction that neither S nor S^c are in $\mathcal{U}$. Let $\mathcal{V}$ be defined as follows:

$$\mathcal{V} := \{V \subseteq \Omega : V \supseteq S \cap U \text{ for some } U \in \mathcal{U}\}$$

We will now show that $\mathcal{V}$ is a filter, contradicting condition 2 for ultrafilters.

1. As $\mathcal{U}$ is a filter, we have $\Omega \in \mathcal{U}$ and therefore as $\Omega \supseteq \Omega \cap S$ it follows that $\Omega \in \mathcal{V}$.
2. If $A \supseteq A' \cap S$ for some $A' \in \mathcal{U}$ and $B \supseteq B' \cap S$ for some $B' \in \mathcal{U}$, then $A \cap B \supseteq A' \cap B' \cap S$ and as $A' \cap B' \in \mathcal{U}$ it follows that $A \cap B \in \mathcal{V}$.
3. If $B \supseteq A$ for some $A \in \mathcal{V}$ then $A \supseteq A' \cap S$ for some $A' \in U$. We have that $B \supseteq A' \cap S$ as well and therefore $B \in \mathcal{V}$.

We therefore have that $\mathcal{V}$ is a filter. We have that $S \supseteq \Omega \cap S$ so $S \in \mathcal{V}$. In addition for all $U \in \mathcal{U}$ we have $U \supseteq U \cap S$. So $\mathcal{U} \subset \mathcal{V}$.
It now suffices to show that $\mathcal{V} \subset P(\Omega)$. Suppose for a contradiction that $\mathcal{V} = P(\Omega)$. It follows that $\emptyset \in \mathcal{V}$ and so $\emptyset \supseteq U \cap S$ for some $U \in \mathcal{U}$. We therefore have that $\emptyset = U \cap S$ and so $U \subseteq S^c$ and $S^c \in \mathcal{U}$, a contradiction. □

Theorem 1.4.6. *Ultrafilters on an infinite set Ω, which have the same moiety elements are equal.*

Proof. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be ultrafilters with the same moiety elements and let $U \in \mathcal{U}_1$. If U is the superset of some moiety then U is also in $\mathcal{U}_2$. If not then $|\Omega \backslash U| = |\Omega|$ and we can therefore construct disjoint moieties M_1 and M_2 such that $M_1 \cup M_2 = \Omega \backslash U$. It therefore follows that $M_1 \cup U$ and $M_2 \cup U$ are moieties and elements of $\mathcal{U}_1$. So we also have that $M_1 \cup U$ and $M_2 \cup U$ are elements of $\mathcal{U}_2$ and therefore their intersection U is in $\mathcal{U}_2$. We therefore have that $\mathcal{U}_1 \subseteq \mathcal{U}_2$ and by symmetry $\mathcal{U}_2 \subseteq \mathcal{U}_1$, so $\mathcal{U}_1 = \mathcal{U}_2$ as required. □

Definition 1.4.7 We say that a set S has the *finite intersection property* if for all finite non-empty subsets $\{s_1, s_2 \dots s_n\}$ of S, we have $s_1 \cap s_2 \cap s_3 \dots s_n \neq \emptyset$.

Theorem 1.4.8 (Ultrafilter Lemma)**.** *Let Ω be a set and let S be a non-empty collection of subsets of Ω with the finite intersection property. Then there is an ultrafilter $\mathcal{U}$ on Ω such that $S \subseteq \mathcal{U}$.*

Proof. The following proof is based on the proof of Theorem 1.8 in [8].
Let $S \subseteq P(\Omega)$ be a non-empty set with the finite intersection property. Let $F(S)$ denote the set of all filters which contain S and do not have the empty set as an element. The set $F(S)$ is partially ordered by $\subseteq$. If $F(S)$ has a maximal element then this element must be an ultrafilter, therefore by Zorn's Lemma it suffices to show that every chain of $F(S)$ has an upper bound.
Consider the set $V_0 := \{U \subseteq \Omega : U \supseteq (s_1 \cap s_2 \cap s_3 \dots \cap s_n)$ for some $s_1, s_2 \dots s_n \in S\}$.

1. We have $\Omega \in V_0$ as S is non-empty and all elements of S are subsets of Ω.
2. If $A, B \in V_0$ then $A \supseteq (s_1 \cap s_2 \dots \cap s_n)$ and $B \subseteq (s_{n+1} \cap s_{n+2} \dots \cap s_m)$ for some $s_1, s_2 \dots s_m \in S$. It follows that $A \cap B \supseteq (s_1 \cap s_2 \dots \cap s_m)$ and thus $A \cap B \in V_0$.
3. If $A \in V_0$ and $A \subseteq B \subseteq \Omega$, then $A \supseteq (s_1 \cap s_2 \dots \cap s_n)$ for some $s_1, s_2 \dots s_n \in S$. Thus $B \supseteq (s_1 \cap s_2 \dots \cap s_n)$ so $B \in V_0$.

We therefore have that V_0 is a filter. By definition $S \subseteq V_0$. As S has the finite intersection property it follows that $\emptyset \notin V_0$. We now have $V_0 \in F(S)$, so V_0 is an upper bound for the empty chain.
Let $(F_i)_{i \in I}$ be a non-empty chain of $F(S)$, where I is an index set. Consider the set $V_1 := \bigcup_{i<\alpha} F_i$.

1. As the non-empty union of filters $\Omega \in V_1$.
2. Let $U_1, U_2 \in V_1$, then $U_1 \in F_{i_1}$ for some i_1 and $U_2 \in F_{i_2}$ for some i_2 and therefore $U_1, U_2 \in \max\{F_{i_1}, F_{i_2}\}$. So $U_1 \cap U_2 \in \max\{F_{i_1}, F_{i_2}\} \subseteq V_1$.
3. Let $U_1 \in V_1$ and let $U_2 \supseteq U_1$, then $U_1 \in F_i$ for some i and therefore $U_2 \in F_i$ so we have $U_2 \in V_1$.

It follows that V_1 is a filter. As a non-empty union of elements of $F(S)$ we have also that $S \subseteq V_1$ and $\emptyset \notin V_1$ so we have $V_1 \in F(S)$. By construction $V_1 \supseteq F_i$ for all $i \in I$ and thus is an upper bound as required. □

Definition 1.4.9 Let Ω and S be sets and $L \subseteq S^{\Omega}$. We say that L has *large oscillation* if the following condition is satisfied: If $n \in \mathbb{N}$ and $\{f_1, f_2 \dots f_n\} \subseteq L$ are distinct and $\{s_1, s_2, s_3 \dots s_n\} \subseteq S$ then there exists $\omega \in \Omega$ such that $(\omega)f_i = s_i$ for all $i < n$.

Theorem 1.4.10. *Let Ω be an infinite set. Then there exists $L \subseteq \{0,1\}^{\Omega}$ such that $|L| = 2^{|\Omega|}$ and L has large oscillation.*

Proof. The following proof is based on the proof of Theorem 2.2 in [8].
Let Ω' be defined by

$$\Omega' := \{(s, S, \phi) : s \subseteq |\Omega| \text{ is finite}, S \subseteq P(s), \phi \in \{0,1\}^S\}.$$

As s is finite we have that $P(P(s))$ is finite, S is finite and $\{0,1\}^S$ is finite. It follows that $|\Omega'| = |\Omega|$ as we have $|\Omega|$ choices for s and finitely many choices for S and ϕ. We may therefore assume without loss of generality that $\Omega = \Omega'$ as $|\Omega| = |\Omega'|$ and Ω'' would give the exact same set as Ω'.
Let $f : P(|\Omega|) \to \{0,1\}^{\Omega}$ be defined by $(\Sigma)f = f_{\Sigma}$ where $f_{\Sigma} : \Omega \to \{0,1\}$ is defined by

$$(s, S, \phi)f_{\Sigma} = \left\{ \begin{array}{ll} (\Sigma \cap s)\phi & \Sigma \cap s \in S \\ 0 & \Sigma \cap s \notin S \end{array} \right\}.$$

Let $L := \operatorname{img}(f)$. To show that $|L| = 2^{|\Omega|}$ it suffices to show that f is injective as $|P(\Omega)| = 2^{|\Omega|}$. Let $\Sigma_1, \Sigma_2 \in P(|\Omega|)$ be distinct. Without loss of generality we have $\Sigma_1 \backslash \Sigma_2 \neq \emptyset$. Let $x \in \Sigma_1 \backslash \Sigma_2$. Let $s := \{x\}$, $S := \{s\}$ and ϕ be such that $(s)\phi = 1$ then

$$(s, S, \phi)f_{\Sigma_1} = (\Sigma_1 \cap s)\phi = (s)\phi = 1 \neq 0 = (\emptyset)\phi = (s \cap \Sigma_2)\phi = (s, S, \phi)f_{\Sigma_2}.$$

It follows that $(\Sigma_1)f \neq (\Sigma_2)f$ and thus f is injective.
It remains to show that L has large oscillation. Let $n \in \mathbb{N}$, $\{f_{A_m} : m < n\} \subseteq L$ be distinct and $\{k_m : m < n\} \subseteq \{0, 1\}$. For each (m_1, m_2) such that $m_1 < m_2 < n$ let $a_{m_1,m_2} \in A_{m_1} \triangle A_{m_2}$.
Let $s := \{a_{m_1,m_2} : m_1 < m_2 < n\}$, $S := \{A_m \cap s : m < n\}$ and let $\phi : S \to \{0, 1\}$ be defined by $(A_m \cap s)\phi = k_m$.
To see that ϕ is well-defined (that no elements of S can be represented as above in two ways) observe that if $A_{m_1} \neq A_{m_2}$ then $a_{m_1,m_2} \in A_{m_1} \triangle A_{m_2}$ and so is in precisely one of $s \cap A_{m_1}$ and $s \cap A_{m_2}$.
We have that $(s, S, \phi) \in \Omega$ and for all $m < n$ we have $(s, S, \phi)f_{A_m} = (s \cap A_m)\phi = k_m$ as required. □

Theorem 1.4.11. *For all infinite sets Ω there are $2^{2^{|\Omega|}}$ ultrafilters on Ω.*

Proof. The following proof is based on the proof of Theorem 2.5 in [8].
Let Ω be an infinite set. We have that any ultrafilter on Ω is an element of $P(P(\Omega))$ and therefore there are at most $|P(P(\Omega))| = 2^{2^{|\Omega|}}$ of them. To show there are at least $2^{2^{|\Omega|}}$ we will construct such a family of ultrafilters.
By Theorem 1.4.10 let $L \subseteq \{0, 1\}^{\Omega}$ be a family of large oscillation such that $|L| = 2^{|\Omega|}$. For $S \in P(L)$ we define $B(S)$ by

$$B(S) := \{(\{0\})f^{-1} : f \in S\} \cup \{(\{1\})f^{-1} : f \in S^c\},$$

where f^{-1} is used to denote preimage. Let $B_1, B_2 \ldots B_k \in B(S)$ then it follows that for $i \in \{1, 2, \ldots k\}$ we have $B_i = (b_i)f_i^{-1}$ for some $f_1, f_2 \ldots f_k \in L$ and $b_1, b_2 \ldots b_k \in \{0, 1\}$. As L has large oscillation it follows that for some $\omega \in \Omega$, $(\omega)f_i = b_i$ for all $i \in \{1, 2 \ldots k\}$, and therefore $\omega \in \bigcap_{i\in\{1,2,\ldots k\}} B_i$. Therefore $B(S)$ has the finite intersection property. By the Ultrafilter Lemma we have that for every $S \in P(L)$ we can extend $B(S)$ to an ultrafilter $\mathcal{U}(S)$.
Suppose for a contradiction that there exist distinct $S_1, S_2, \in P(L)$ such that $\mathcal{U}(S_1) = \mathcal{U}(S_2)$. Then without loss of generality there exists $f \in S_1 \backslash S_2$. We have that $(\{0\})f^{-1} \in \mathcal{U}(S_1)$ is the complement of $(\{1\})f^{-1} \in \mathcal{U}(S_2)$. So because $\mathcal{U}(S_1) = \mathcal{U}(S_2)$ we have that $(\{0\})f^{-1} \cap (\{1\})f^{-1} = \emptyset \in \mathcal{U}(S_1)$, a contradiction. It follows that $\mathcal{U}(S)$ for $S \in P(L)$ are distinct ultrafilters and there is $|P(L)| = 2^{2^{|\Omega|}}$ of them as required. □

This theorem gives an interesting corollary, that for any infinite set Ω, there are $2^{2^{|\Omega|}}$ topologies on Ω. This follows as filters can be extended to topologies by adding the empty set.

Chapter 2

Topological Groups and Commutators

In this chapter we will be primarily focusing on $\mathrm{Sym}(\mathbb{N})$. We will be viewing this group as a topological group and will use its topological properties to show that all its elements can be written as commutators. We will then generalise this property to all infinite symmetric groups.

2.1 Infinite Permutations

Theorem 2.1.1. *If Ω is an infinite set and $f, g \in \mathrm{Sym}(\Omega)$, then f and g are conjugate if and only if they have the same disjoint cycle shape.*

Proof. ($\Leftarrow$) Let $C_{f,i}$ be the set of i-cycles of f and similarly $C_{g,i}$ be the set of i-cycles of g. We have that $|C_{f,i}| = |C_{g,i}|$ for all $i \in \mathbb{N}_0 \cup \{\aleph_0\}$. Label their elements such that

$$C_{f,i} = \{C_{f,i,j} : j \in |C_{f,i}|\}, \qquad C_{g,i} = \{C_{g,i,j} : j \in |C_{g,i}|\}.$$

Let $C_{f,i,j} := (C_{f,i,j,0}, C_{f,i,j,1}, \dots C_{f,i,j,i-1})$ if $i \neq \aleph_0$ and $C_{f,i,j} = (\dots C_{f,i,j,-1}, C_{f,i,j,0}, C_{f,i,j,1} \dots)$ otherwise. Similarly let $C_{g,i,j} := (C_{g,i,j,0}, C_{g,i,j,1} \dots C_{g,i,j,i-1})$ if $i \neq \aleph_0$ and $C_{g,i,j} = (\dots C_{g,i,j,-1}, C_{g,i,j,0}, C_{g,i,j,1} \dots)$ otherwise. Note that the domains of the disjoint cycles of f or g partition Ω. We can therefore define a bijection $h : \Omega \to \Omega$ by $(C_{f,i,j,k})h = (C_{g,i,j,k})$. We now have that $f = hgh^{-1}$ as required.
($\Rightarrow$) Suppose that $h^{-1}gh = f$ for some $h \in \mathrm{Sym}(\Omega)$. Let $C_f := (\dots c_{-1}, c_0, c_1 \dots)$ be a cycle of f, we have $h^{-1}C_f h = (\dots (c_{-1})h, (c_0)h, (c_1)h \dots)$. Similarly if $C_f = (c_1 \dots c_k)$ is a cycle of f. we have $h^{-1}C_f h = ((c_1)h, (c_2)h \dots (c_k)h)$.
We therefore have that all disjoint cycles in f have a unique corresponding cycle in g and therefore g has the same number of cycles of each length as f and so g has the same disjoint cycle shape as f. □

The desired result for this chapter will take some time to prove, however there is a similar weaker result which can be shown with relatively little effort. In addition this proof gives us a way to construct the elements that we are commutating.

Theorem 2.1.2. *If $f \in \mathrm{Sym}(\mathbb{N}_0)$ and f has finite support, then f can be written in the form*

$$f = [g, h] = g^{-1}h^{-1}gh$$

where $g, h \in \mathrm{Sym}(\mathbb{N}_0)$.

Proof. As f has finite support we have that there exists $k \in \mathbb{N}_0$ such that $(i)f = i$ for all $i \geq k$. For $n \in \mathbb{N}_0$ we define $q_n \in \mathbb{N}_0$ and $r_n \in \{0, 1 \dots 2k-1\}$ by the expression $n = 2kq_n + r_n$ (quotient and remainder when dividing by 2k).
Let $g, h \in \mathrm{Sym}(\mathbb{N}_0)$ be defined as follows:

$$(n)g = \left\{ \begin{array}{ll} 2q_n k + (r_n)f^{-1} & r_n < k \\ n & \textit{otherwise} \end{array} \right\}, \qquad (n)h = \left\{ \begin{array}{ll} n-k & k \leq n < 2k \\ n+2k & r_n < k \\ n-2k & r_n \geq k \text{ and } q_n > 0 \end{array} \right\}.$$

Note that g, h have the following inverses and are therefore bijections:

$$(n)g^{-1} = \left\{ \begin{array}{ll} 2q_n k + (r_n)f & r_n < k \\ n & \text{otherwise} \end{array} \right\}, \qquad (n)h^{-1} = \left\{ \begin{array}{ll} n+k & n < k \\ n-2k & r_n < k \text{ and } q_n > 0 \\ n+2k & r_n \geq k \end{array} \right\}.$$

We now show that $f = [g, h]$.

Case 1: $n < k$

$$\begin{aligned}(n)[g,h] &= ((((n)g^{-1})h^{-1})g)h \\ &= ((((r_n)g^{-1})h^{-1})g)h \\ &= (((r_n)f + k)g)h \\ &= ((r_n)f + k)h \\ &= (r_n)f = (n)f.\end{aligned}$$

Case 2: $q_n > 0$ and $r_n < k$

$$\begin{aligned}(n)[g,h] &= ((((n)g^{-1})h^{-1})g)h \\ &= ((((2q_nk + r_n)g^{-1})h^{-1})g)h \\ &= (((2q_nk + (r_n)f)h^{-1})g)h \\ &= ((2(q_n - 1)k + (r_n)f)g)h \\ &= 2q_nk + r_n = n = (n)f.\end{aligned}$$

Case 3: $k \leq r_n$

$$\begin{aligned}(n)[g,h] &= ((((n)g^{-1})h^{-1})g)h \\ &= (((n)h^{-1})g)h \\ &= ((n + 2k)g)h \\ &= (n + 2k)h \\ &= n = (n)f.\end{aligned}$$

□

2.2 Constructing a Topology on an Infinite Symmetric Group

In this section we will be constructing a completely metrisable topology on Sym($\mathbb{N}$), by first constructing one on the full transformation monoid and then considering the subspace topology. In doing this we will use the more general results concerning topologies and metric spaces given in the previous chapter.

Definition 2.2.1 The countably infinite product of the discrete topology on $\mathbb{N}$ will be denoted by $\mathscr{T}$. This topology is defined on the set $\mathbb{N}^{\mathbb{N}}$. The following notations will also be used:

$$\mathbb{N}^{<\mathbb{N}} := \{\sigma : \{1, 2 \ldots n\} \to \mathbb{N} : n \in \mathbb{N}\}, \qquad [\sigma] := \{f \in \mathbb{N}^{\mathbb{N}} : f|_{\mathrm{dom}(\sigma)} = \sigma\}, \qquad B := \{[\sigma] : \sigma \in \mathbb{N}^{<\mathbb{N}}\}.$$

Theorem 2.2.2. *The set B above is a basis for $\mathscr{T}$.*

Proof. We have that $B_o := \{U_1 \times U_2 \ldots \times U_k \times \mathbb{N} \times \mathbb{N} \times \ldots : k \in \mathbb{N},\ U_i \subseteq \mathbb{N} \text{ for all } i \in \{1, 2 \ldots k\}\}$ is a basis for $\mathscr{T}$ by the definition of an infinite product topology.
As $B \subseteq B_o$ it suffices to show that for all $U \in B_o$ there exists $G_U \subseteq B$ such that $U = \cup G_U$. Let U be given by

$$U = U_1 \times U_2 \ldots \times U_k \times \mathbb{N} \times \mathbb{N} \times \ldots$$

Let $G_U := \{[\sigma] \in B : \mathrm{dom}(\sigma) = \{1, 2 \ldots k\}$ and for all $i \in \mathrm{dom}(\sigma)$ we have $(i)\sigma \in U_i\}$. We now show that $U = \cup G_U$.
($\subseteq$) Let $f \in U$. We have that $(i)f \in U_i$ for all $i \in \{1, 2 \ldots k\}$. So if we let $\sigma_f := f|_{\{1,2\ldots k\}} \in \mathbb{N}^{<\mathbb{N}}$, we have that $f \in [\sigma_f] \in G_U$ so $f \in \cup G_U$.
($\supseteq$) Let $f \in \cup G_U$. There exists $[\sigma_f] \in G_U$ such that $f \in [\sigma_f]$. It follows that $(i)f = (i)\sigma_f \in U_i$ for all $i \in \{1, 2 \ldots k\}$ so $f \in U$. □

Theorem 2.2.3. *If $d : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{R}$ is defined by*

$$d(f, g) = \left\{ \begin{array}{ll} 0 & f = g \\ \frac{1}{\min\{i \in \mathbb{N} : (i)f \neq (i)g\}} & f \neq g \end{array} \right\}$$

then the tuple $(\mathbb{N}^{\mathbb{N}}, d)$ is a complete metric space which induces the topology $\mathscr{T}$, and thus $\mathscr{T}$ is completely metrisable.

Proof. We first show that d is a metric. The non-negativity, identity of indiscernibles and symmetry conditions follow immediately from the definition.
Triangle inequality: Let $f, g, h \in \mathbb{N}^{\mathbb{N}}$. First notice that if $j = \min\{i \in \mathbb{N} : (i)f \neq (i)h\}$ then either $(j)f \neq (j)g$ or $(j)g \neq (j)h$. It therefore follows that

$$\min\{i \in \mathbb{N} : (i)f \neq (i)h\} \geq \min\{i \in \mathbb{N} : (i)f \neq (i)g \text{ or } (i)g \neq (i)h\} = \min\{\min\{i \in \mathbb{N} : (i)f \neq (i)g\}, \min\{i \in \mathbb{N} : (i)g \neq (i)h\}\}$$
$$\implies \frac{1}{d(f,h)} \geq \min\{\frac{1}{d(f,g)}, \frac{1}{d(g,h)}\} = \frac{1}{\max\{d(f,g), d(g,h)\}}$$
$$\implies d(f,h) \leq \max\{d(f,g), d(g,h)\} \leq d(f,g) + d(g,h).$$

We next show that $(\mathbb{N}^{\mathbb{N}}, d)$ is complete.
Let $S := (f_1, f_2, f_3 \ldots)$ be a Cauchy sequence.
Claim: for all $i \in \mathbb{N}$ there is a minimal $M_i \in \mathbb{N}$ such that for all $j \in \{1, 2 \ldots i\}$ and all $n \geq M_i$

$$(j)f_{M_i} = (j)f_n.$$

Proof of Claim: Let $i \in \mathbb{N}$. As S is Cauchy we have that for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $d(f_n, f_m) < \varepsilon$. By choosing $\varepsilon = \frac{1}{i+1}$ we have that for all $i \in \mathbb{N}$, there exists N_i such that for all $n, m \geq N_i$ we have $d(f_n, f_m) < \frac{1}{i+1}$. It follows that $d(f_{N_i}, f_n) < \frac{1}{i+1}$ for all $n \geq N_i$, and so $(j)f_{N_i} = (j)f_n$ for all $n \geq N_i$ and $j \in \{1, 2 \ldots i\}$.

As there exists a natural number N_i with the desired property and the natural numbers are well ordered there must exist a minimal such number M_i with this property. $\square$

Define $l : \mathbb{N} \to \mathbb{N}$ by $(i)l = (i)f_{M_i}$. It suffices to show that S converges to l. We have from the definition of M_i that for all $i \in \mathbb{N}$ we have $M_i \leq M_{i+1}$. For all $i \in \mathbb{N}$ we have $M_i \leq M_{i+1}$, so for all $i \in \mathbb{N}$ we have $(i)f_{M_{i+1}} = (i)f_{M_i}$. It follows that for all $j > i$ we have $(i)f_{M_i} = (i)f_{M_j}$.
For all $n \geq M_i$ and all $j \in \{1, 2 \dots i\}$ we have $(j)l = (j)f_{M_j} = (j)f_{M_i} = (j)f_n$ and therefore $d(l, f_n) < \frac{1}{i}$.
We have just shown that for all $i \in \mathbb{N}$ there exists $M_i \in \mathbb{N}$ such that for all $n \geq M_i$ we have $d(l, f_n) < \frac{1}{i}$, so S converges to l as required. Finally we show that $\mathscr{T}$ is induced by d. By definition, the topology induced by d is given by $\langle \{\{g \in \mathbb{N}^{\mathbb{N}} : d(g, f) < \varepsilon\} : f \in \mathbb{N}^{\mathbb{N}}, \varepsilon > 0\} \rangle_T$. We have that

$$\begin{aligned}
\langle \{\{g \in \mathbb{N}^{\mathbb{N}} : d(g, f) < \varepsilon\} : f \in \mathbb{N}^{\mathbb{N}}, \varepsilon > 0\} \rangle_T &= \langle \{\{g \in \mathbb{N}^{\mathbb{N}} : (i)g = (i)f \text{ for all } i \in \{1, 2 \dots (\lceil \frac{1}{\varepsilon} \rceil - 1)\}\} : f \in \mathbb{N}^{\mathbb{N}}, \varepsilon > 0\} \rangle_T \\
&= \langle \{\{g \in \mathbb{N}^{\mathbb{N}} : (j)g = (j)f \text{ for all } j \in \{1, 2 \dots i\}\} : f \in \mathbb{N}^{\mathbb{N}}, i \in \mathbb{N}\} \rangle_T \\
&= \langle \{\{g \in \mathbb{N}^{\mathbb{N}} : (j)g = (j)\sigma \text{ for all } j \in \operatorname{dom}(\sigma)\} : \sigma \in \mathbb{N}^{<\mathbb{N}}\} \rangle_T \\
&= \langle \{\{g \in \mathbb{N}^{\mathbb{N}} : g|_{\operatorname{dom}(\sigma)} = \sigma\} : \sigma \in \mathbb{N}^{<\mathbb{N}}\} \rangle_T \\
&= \langle \{[\sigma] : \sigma \in \mathbb{N}^{<\mathbb{N}}\} \rangle_T = \mathscr{T}.
\end{aligned}$$

$\square$

Theorem 2.2.4. *The infinite symmetric group* Sym($\mathbb{N}$) *is a* G_δ *subset of* $(\mathbb{N}^{\mathbb{N}}, \mathscr{T})$.

Proof. Let I denote the set of all injective functions from $\mathbb{N}$ to $\mathbb{N}$. We have that

$$I = \bigcap_{i \neq j} \{g \in \mathbb{N}^{\mathbb{N}} : (i)g \neq (j)g\}.$$

We now show that for all $i \neq j$ we have $\{g \in \mathbb{N}^{\mathbb{N}} : (i)g \neq (j)g\}$ is open. Let $i, j \in \mathbb{N}$ be such that $i \neq j$, and let $g \in \mathbb{N}^{\mathbb{N}}$ be such that $(i)g \neq (j)g$. For $h \in B(g, \frac{1}{i+j})$ we have that $(i)h = (i)g \neq (j)g = (j)h$. It follows that $B(g, \frac{1}{i+j}) \subseteq \{g \in \mathbb{N}^{\mathbb{N}} : (i)g \neq (j)g\}$. As g was arbitrary it follows that $\{g \in \mathbb{N}^{\mathbb{N}} : (i)g \neq (j)g\}$ is open.
Let S denote the set of all surjective functions from $\mathbb{N}$ to $\mathbb{N}$. We have that

$$S = \bigcap_{k \in \mathbb{N}} \{g \in \mathbb{N}^{\mathbb{N}} : k \in \operatorname{img}(g)\}.$$

We now show that for all $k \in \mathbb{N}$, the set $\{g \in \mathbb{N}^{\mathbb{N}} : k \in \operatorname{img}(g)\}$ is open. Let $k \in \mathbb{N}$ and let g be such that $k \in \operatorname{img}(g)$. Let $k_g \in \mathbb{N}$ be such that $(k_g)g = k$. It follows that for $h \in B(g, \frac{1}{k_g+7})$, we have $(k_g)h = (k_g)g = k$ and thus $h \in \{g \in \mathbb{N}^{\mathbb{N}} : k \in \operatorname{img}(g)\}$. Therefore $B(g, \frac{1}{k_g+7}) \subseteq \{g \in \mathbb{N}^{\mathbb{N}} : k \in \operatorname{img}(g)\}$. As g was arbitrary it follows that $\{g \in \mathbb{N}^{\mathbb{N}} : k \in \operatorname{img}(g)\}$ is open.
We now have S and I as a countable intersection of open sets. It follows that $\operatorname{Sym}(\mathbb{N}) = I \cap S$ is a countable intersection of open sets as required. $\square$

Theorem 2.2.5. *Let* $\mathscr{T}_s$ *be the subspace topology of* Sym($\mathbb{N}$) *in* $(\mathbb{N}^{\mathbb{N}}, \mathscr{T})$. *The topological space* $(\operatorname{Sym}(\mathbb{N}), \mathscr{T}_s)$ *is completely metrisable.*

Proof. By Theorems 1.1.1, 2.2.4 and 1.1.4 we have that $(\mathbb{N}^{\mathbb{N}}, \mathscr{T})$ is completely metrisable, Sym($\mathbb{N}$) is G_δ in $(\mathbb{N}^{\mathbb{N}}, \mathscr{T})$ and G_δ sets of a completely metrisable topology equipped with the subspace topology are completely metrisable. It therefore follows that $(\operatorname{Sym}(\mathbb{N}), \mathscr{T}_s)$ is completely metrisable. $\square$

Theorem 2.2.6. *The triple* $(\operatorname{Sym}(\mathbb{N}), \circ, \mathscr{T}_s)$ *is a topological group (where* $\circ$ *represents composition of functions).*

Proof. For $\sigma \in \mathbb{N}^{<\mathbb{N}}$ let $[\sigma]_s := \{f \in \operatorname{Sym}(\mathbb{N}) : f|_{\operatorname{dom}(\sigma)} = \sigma\}$. As B is a basis for $\mathscr{T}$ we have that $B_s := \{b \cap \operatorname{Sym}(\mathbb{N}) : b \in B\} = \{[\sigma]_s : \sigma \in \mathbb{N}^{<\mathbb{N}}\}$ is a basis for $\mathscr{T}_s$. Let $\phi_i : \operatorname{Sym}(\mathbb{N}) \to \operatorname{Sym}(\mathbb{N})$ be defined by $(f)\phi_i = f^{-1}$. If suffices to show that $(b)\circ^{-1}$ and $(b)\phi_i^{-1} = (b)\phi_i$ are open for all $b \in B_s$.
Let $b = [\sigma]_s \in B_s$ and let $\operatorname{dom}(\sigma) := \{1, 2 \dots k\}$. We have

$$(f, g) \in (b)\circ^{-1} \implies fg \in b = [\sigma]_s \implies ((i)f)g = (i)\sigma \quad \text{for all } i \in \{1, 2 \dots k\}.$$

Let $U_{f,g} := [f|_{\{1,2\dots k\}}]_s \times [g|_{\{1,2\dots \max\{(i)f : i \in \{1,2\dots k\}\}\}}]_s$. We have that $(f, g) \in U_{f,g}$ and $U_{f,g}$ is open.
Let $(f_2, g_2) \in U_{f,g}$.

$$\begin{aligned}
&((i)f_2)g_2 = ((i)f)g_2 \quad \text{for all } i \in \{1, 2 \dots k\} \quad (\text{as } f_2|_{\{1,2\dots k\}} = f|_{\{1,2\dots k\}}) \\
&\implies ((i)f_2)g_2 = ((i)f)g \quad \text{for all } i \in \{1, 2 \dots k\} \quad (\text{as } g_2|_{\{1,2\dots \max\{(i)f : i \in \{1,2\dots k\}\}\}} = g|_{\{1,2\dots \max\{(i)f : i \in \{1,2\dots k\}\}\}}) \\
&\implies ((i)f_2)g_2 = (i)\sigma \text{ for all } i \in \{1, 2 \dots k\} = \operatorname{dom}(\sigma) \\
&\implies (f_2, g_2) \in ([\sigma]_s)\circ^{-1} = (b) \circ^{-1}.
\end{aligned}$$

So we have that for all $(f,g) \in (b)\circ^{-1}$ there is an open set $U_{f,g}$ such that $(f,g) \in U_{f,g} \subseteq (b)\circ^{-1}$ and therefore $(b)\circ^{-1}$ is open and $\circ$ is continuous.

$$f \in (b)\phi_i \implies f^{-1} \in [\sigma]_s \implies (i)f^{-1} = (i)\sigma \text{ for all } i \in \{1, 2 \dots k\} \implies i = ((i)\sigma)f \text{ for all } i \in \{1, 2 \dots k\}.$$

Let $U_f := [f|_{\{1,2\dots\max(\text{img}(\sigma))\}}]_s$. We have that $f \in U_f$ and U_f is open. It therefore suffices to show that $U_f \subseteq (b)\phi_i$. Let $f_2 \in U_f$.

$$\begin{aligned} &((i)\sigma)f_2 = ((i)\sigma)f \text{ for all } i \in \{1, 2 \dots k\} \quad (\text{as } f_2|_{\{1,2\dots\max(\text{img}(\sigma))\}} = f|_{\{1,2\dots\max(\text{img}(\sigma))\}}) \\ &\implies ((i)\sigma)f_2 = i \text{ for all } i \in \{1, 2 \dots k\} \\ &\implies (i)\sigma = (i)f_2^{-1} \text{ for all } i \in \{1, 2 \dots k\} \\ &\implies f_2^{-1} \in [\sigma]_s \implies (f_2^{-1})\phi_i \in ([\sigma]_s)\phi_i = (b)\phi_i \implies f_2 \in (b)\phi_i. \end{aligned}$$

□

2.3 Comeagre Conjugacy Class

For the rest of this chapter we will construct a comeagre conjugacy class of $\text{Sym}(\mathbb{N})$ and use it, with the Baire Catagory Theorem, to show the desired result that all elements of $\text{Sym}(\mathbb{N})$ are commutators.

Definition 2.3.1 Let the conjugacy class C be defined by

$$C := \{f \in \text{Sym}(\mathbb{N}) : f \text{ has infinitely many cycles of every finite length but has no cycles of infinite length}\}.$$

Note that this actually is a conjugacy class by Theorem 2.1.1.

Theorem 2.3.2. *Let $I := (i_1, i_2, \dots i_k)$ and $N := (n_1, n_2, \dots n_k)$ be finite sequences of natural numbers with no repeats. There exists $\sigma \in \text{Sym}(\mathbb{N})$ and $r > 0$ such that $(n_j)f = i_j$ for all $f \in B(\sigma, r)$ and for all $j \in \{1, 2 \dots k\}$.*

Proof. As both I and N are finite, we have that $|\mathbb{N}\backslash I| = |\mathbb{N}| = |\mathbb{N}\backslash N|$. Therefore there exists a bijection $\phi : \mathbb{N}\backslash N \to \mathbb{N}\backslash I$. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be given by

$$(n)\sigma = \left\{ \begin{array}{ll} i_j & \text{if } n = n_j \text{ for some } j \in \{1, 2, \dots k\} \\ (n)\phi & \text{otherwise} \end{array} \right\}.$$

By construction $\sigma \in \text{Sym}(\mathbb{N})$. Let $r := \frac{1}{\max(N)}$ and $f \in B(\sigma, r)$.

$$\begin{aligned} d(\sigma, f) < \frac{1}{\max(N)} &\implies \frac{1}{\min(\{n \in \mathbb{N} : (n)\sigma \neq (n)f\})} < \frac{1}{\max(N)} \\ &\implies \max(N) < \min(\{n \in \mathbb{N} : (n)\sigma \neq (n)f\}) \\ &\implies (n_j)f = (n_j)\sigma \text{ for all } j \in \{1, 2, \dots k\} \\ &\implies (n_j)f = i_j \text{ for all } j \in \{1, 2, \dots k\}. \end{aligned}$$

□

Theorem 2.3.3. *The sets $C_{1,i}$ defined by*

$$C_{1,i} = \{f \in \text{Sym}(\mathbb{N}) : f \textit{ has a cycle of infinite length with } i \textit{ in its domain}\}$$

are nowhere dense for all $i \in \mathbb{N}$

Proof. Let $i \in \mathbb{N}$, $g \in \text{Sym}(\mathbb{N})$, $r_1 > 0$ and $g_r := g|_{\{1,2\dots(\lceil \frac{1}{r_1} \rceil - 1)\}}$. We have $B(g, r_1) = \{f \in \text{Sym}(\mathbb{N}) : f|_{\{1,2\dots(\lceil \frac{1}{r_1} \rceil - 1)\}} = g_r\}$.
By Theorem 1.1.6 it suffices to find $\sigma \in \text{Sym}(\mathbb{N})$ and $r > 0$ such that $B(\sigma, r) \subseteq B(g, r_1)\backslash C_{1,i}$.

$$m_1 := \min\{j \in \mathbb{N}_0 : (i)g^j \notin \text{dom}(g_r)\}, \qquad m_2 := \min\{j \in \mathbb{N}_0 : (i)g^{-j} \notin \text{img}(g_r)\}.$$

Note that if m_1 and m_2 don't exist then $B(g, r_1) \subseteq B(g, r_1)\backslash C_{1,i}$ and we are done.
By Theorem 2.3.2 let σ be a bijection satisfying

$$(j)\sigma = (j)g \text{ for all } j \in \text{dom}(g_r), \qquad ((i)g^{m_1})\sigma = (i)g^{-m_2}$$

and let $r > 0$ be such that for all $f \in B(\sigma, r)$ we have

$$(j)\sigma = (j)f \text{ for all } j \in \text{dom}(g_r), \qquad ((i)g^{m_1})\sigma = ((i)g^{m_1})f.$$

We now have that $|\text{orb}_{\{f\}}(i)| \leq m_1 + m_2 < \infty$ for all $f \in B(\sigma, r)$, and therefore $f \notin C_{1,i}$. So $B(\sigma, r) \subseteq B(g, r_1)\backslash C_{1,i}$ as required. □

Theorem 2.3.4. *The sets $C_{2,i,j}$ defined by*

$$C_{2,i,j} = \{f \in \mathrm{Sym}(\mathbb{N}) : f \text{ has } i \text{ cycles of length } j\}$$

are nowhere dense for all $i \in \mathbb{N}_0$ and $j \in \mathbb{N}$.

Proof. Let $i \in \mathbb{N}_0$ and $j \in \mathbb{N}$. Let $g \in \mathrm{Sym}(\mathbb{N})$, $r_1 > 0$ and $g_r := g|_{\{1,2\ldots(\lceil \frac{1}{r_1}\rceil -1)\}}$. We have $B(g, r_1) = \{f \in \mathrm{Sym}(\mathbb{N}) : f|_{\{1,2\ldots(\lceil \frac{1}{r_1}\rceil -1)\}} = g_r\}$. By Theorem 1.1.6 it suffices to find $\sigma \in \mathrm{Sym}(\mathbb{N})$ and $r > 0$ such that $B(\sigma, r) \subseteq B(g, r_1)\backslash C_{2,i,j}$. Let $m := \max(\mathrm{dom}(g_r) \cup \mathrm{img}(g_r)) + 1$. By Theorem 2.3.2 let $\sigma \in \mathrm{Sym}(\mathbb{N})$ be such that

$$\begin{aligned}
(k)\sigma &= (k)g \quad \text{for all } k \in \mathrm{dom}(g_r)\\
(m+1)\sigma &= (m+2), (m+2)\sigma = (m+3)\ldots(m+j)\sigma = (m+1),\\
(m+j+1)\sigma &= (m+j+2)\ldots,(m+2j)\sigma = (m+j+1),\\
(m+2j+1)\sigma &= (m+2j+2)\ldots,(m+3j)\sigma = (m+2j+1),\\
&\vdots\\
(m+ij+1)\sigma &= (m+ij+2)\ldots(m+(i+1)j)\sigma = (m+ij+1)
\end{aligned}$$

and let $r > 0$ be such that for all $f \in B(\sigma, r)$ we have

$$(k)f = (k)\sigma \quad \text{for all } k \in \{1, 2\ldots(m+(i+1)j)\}.$$

All $f \in B(\sigma, r)$ have at least $(i+1)$ cycles of length j and therefore $f \notin C_{2,i,j}$. However $B(\sigma, r) \subseteq B(x_1, r_1)$ so we have $B(\sigma, r) \subseteq B(x_1, r_1)\backslash C_{2,i,j}$ as required. □

Theorem 2.3.5. *The conjugacy class C is comeagre.*

Proof. Let C_1 and C_2 be defined by:

$C_1 := \{f \in \mathrm{Sym}(\mathbb{N}) : f$ has a cycle of infinite length$\}$, $C_2 := \{f \in \mathrm{Sym}(\mathbb{N}) : f$ has finitly many cycles of some finite length $\}$.

Let $C_{2,i}$ be defined by

$$C_{2,i} := \{f \in \mathrm{Sym}(\mathbb{N}) : \text{f has i cycles of some finite length }\}.$$

Notice that we now have the following equalities:

$$C^c = C_1 \cup C_2, \quad C_1 = \bigcup_{i\in\mathbb{N}} C_{1,i}, \quad C_2 = \bigcup_{i\in\mathbb{N}_0} C_{2,i}, \quad C_{2,i} = \bigcup_{j\in\mathbb{N}} C_{2,i,j}.$$

We therefore have that

$$C^c = C_1 \cup C_2 = (\bigcup_{i\in\mathbb{N}} C_{1,i}) \cup (\bigcup_{i\in\mathbb{N}_0} C_{2,i}) = (\bigcup_{i\in\mathbb{N}} C_{1,i}) \cup (\bigcup_{i\in\mathbb{N}_0} (\bigcup_{j\in\mathbb{N}} C_{2,i,j})).$$

By Theorems 2.3.3 and 2.3.4 we have that all the $C_{1,i}$ and $C_{2,i,j}$ are nowhere dense. Thus C^c is a countable union of nowhere dense sets and is therefore meagre. □

Theorem 2.3.6. *We have $CC = \mathrm{Sym}(\mathbb{N})$ and in particular all functions $f \in \mathrm{Sym}(\mathbb{N})$ can be written in the form*

$$f = [g, h] = g^{-1}h^{-1}gh$$

where $g, h \in \mathrm{Sym}(\mathbb{N})$.

Proof. Let $f \in \mathrm{Sym}(\mathbb{N})$. By Theorems 2.3.5 and 1.1.9, we have that C is comeagre, and right multiplication by f is a homeomorphism. As homomorphisms preserve closures, interiors and unions it follows that $(C)f$ is comeagre. Therefore there exist nowhere dense sets $(N_{1,i})_{i\in\mathbb{N}}$ and $(N_{2,i})_{i\in\mathbb{N}}$ such that $C^c = \cup_{i\in\mathbb{N}} N_{1,i}$ and $((C)f)^c = \cup_{i\in\mathbb{N}} N_{2,i}$. It follows that $((C)f \cap C)^c = \cup_{i\in\mathbb{N}}(N_{1,i} \cup N_{2,i})$ and so $(C)f \cap C$ is comeagre. So by the Baire Catagory Theorem we have $C \cap (C)f \neq \emptyset$. For $x \in C \cap (C)f$, there exist $f_1, f_2 \in C$ such that $f_1 = x = f_2 f$. It follows that $f = f_2^{-1} f_1$. We have that C is closed under taking inverses as the disjoint cycles of the inverse of a permutation are the same but reversed.
It therefore follows that f_2^{-1} is conjugate to f_2, so $f_2^{-1} \in C$. Therefore we have that $f = f_2^{-1} f_1 \in CC$.
As $f_1, f_2 \in C$ we have that there exists $h \in \mathrm{Sym}(\mathbb{N})$ such that $f_1 = h^{-1} f_2 h$. Let $g := f_2$, we now have

$$[g, h] = g^{-1}h^{-1}gh = f_2^{-1}h^{-1}f_2 h = f_2^{-1} f_1 = f.$$

□

We now have that every element of $\mathrm{Sym}(\mathbb{N})$ is a commutator. It remains to show for any infinite set Ω that any element of $\mathrm{Sym}(\Omega)$ can be written as a commutator.

Theorem 2.3.7. *Let Ω be an infinite set. Then all functions $f \in \mathrm{Sym}(\Omega)$ can be written in the form:*

$$f = [g, h] = g^{-1}h^{-1}gh$$

where $g, h \in \mathrm{Sym}(\Omega)$. In addition the conjugacy class

$$C_\Omega := \{f \in \mathrm{Sym}(\Omega) : f \text{ has } |\Omega| \text{ cycles of every finite length and no cycles of infinite length}\}$$

satisfies $C_\Omega C_\Omega = \mathrm{Sym}(\Omega)$.

Proof. If $|\Omega| = \aleph_0$ then we are done by Theorem 2.3.6. Otherwise let $f \in \mathrm{Sym}(\Omega)$. For any point $p \in \Omega$ we have that $|\operatorname{orb}_{\{f\}}(p)|$ is countable. Let P_1 be the partition of Ω into the orbits of points under f. We must have that $|P_1| = |\Omega|$ as if $|P_1| > |\Omega|$ we would have that $|\cup P_1| > |\Omega| = |\cup P_1|$, and if $|P_1| < |\Omega|$ then $|\cup P_1| \leq |P_1| \times \aleph_0 \leq \max\{|P_1|^2, \aleph_0\} = \max\{|P_1|, \aleph_0\} < |\Omega| = |\cup P_1|$. We can therefore index P_1 as $P_1 = \{S_i : i \in |\Omega|\}$. Define an equivalence relation by $S_i \sim S_j \iff i = \alpha + a$ and $j = \alpha + b$ for some $a, b \in \mathbb{N}_0$ and limit ordinal α.
Let P_2 be the partition of P_1 into equivalence classes by this relation. For the same reasons as P_1 we have that $|P_2| = |\Omega|$. Let $P := \{\cup x : x \in P_2\}$. We therefore have that $P = \{P_i : i \in |\Omega|\}$ is a partition of Ω into countably infinite sets such that for all $p \in \Omega$ we have that $p \in P_i \iff (p)f \in P_i$. We can therefore consider f as an element of $\prod_{i \in |\Omega|} \mathrm{Sym}(P_i)$. As each P_i is countably infinite it follows from Theorem 2.3.6 that, for all $i \in |\Omega|$, the permutation $f|_{P_i}$ can be written as $f_{1i}f_{2i}$ for some $f_{1i}, f_{2i} \in \mathrm{Sym}(P_i)$ with infinitely many cycles of all finite lengths and none of infinite length. From these we can define f_1, f_2 by:

$$(p)f_1 := (p)f_{1i} \text{ were } i \text{ is the index of the unique } P_i \text{ containing } p,$$

$$(p)f_2 := (p)f_{2i} \text{ were } i \text{ is the index of the unique } P_i \text{ containing } p.$$

We have that $f|_{P_i} = f_1 f_2|_{P_i}$ for all $i \in |\Omega|$ and thus $f_1 f_2 = f$. By the construction of f_1 and f_2, they have $|\Omega| \times \aleph_0 = |\Omega|$ cycles of all finite lengths and $|\Omega| \times 0 = 0$ cycles of infinite length so $f_1, f_2 \in C_\Omega$ and $f \in C_\Omega C_\Omega$. It remains to show that f is a commutator. As $f_1, f_2 \in C_\Omega$ it follows that there exists $h \in \mathrm{Sym}(\Omega)$ such that $h^{-1}f_1^{-1}h = f_2$. Let $g := f_1^{-1}$. We now have

$$[g, h] = g^{-1}h^{-1}gh = f_1 h^{-1} f_1^{-1} h = f_1 f_2 = f.$$

□

Chapter 3

Generating the Infinite Symmetric Groups

In this chapter we will look at various ways of generating infinite symmetric groups.

3.1 The Bergman Property

Definition 3.1.1 A semigroup S is said to have the *Bergman Property* if, for all $U \subseteq S$ such that $\langle U \rangle_S = S$, there exists a natural number n such that $\bigcup_{i=1}^{n} U^i = S$.

An equivalent way to view this concept is that the semigroup's Cayley graph will always have bounded diameter. Note that all finite semigroups have the semigroup Bergman Property. In this section we will show that given an infinite set Ω the group $\text{Sym}(\Omega)$ has the Bergman Property. This is an unusual property among infinite semigroups.

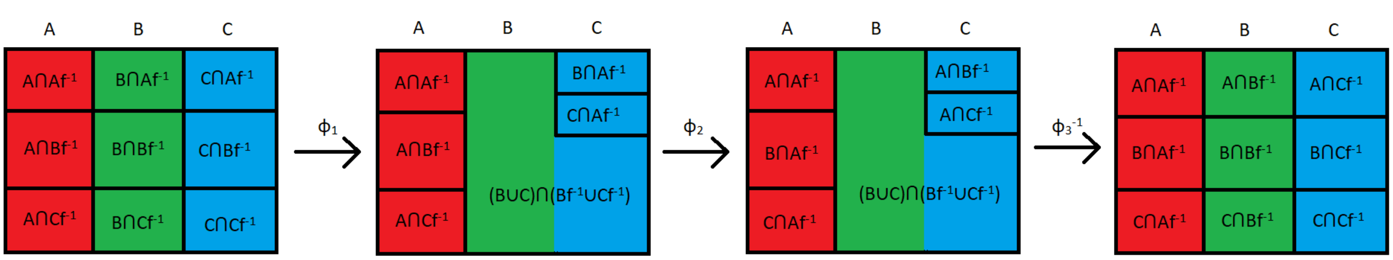
Example 3.1.2 For $S \in \{(\mathbb{N},+),(\mathbb{Z},+),(\mathbb{Q},+),(\mathbb{R},+),(\mathbb{C},+),(\mathbb{Q}\backslash\{0\},\times),(\mathbb{R}\backslash\{0\},\times),(\mathbb{C}\backslash\{0\},\times)\}$ there exists $U \subseteq S$ such that $\langle U \rangle_S = S$ and $\bigcup_{i=1}^{n} U^i \neq S$ for all $n \in \mathbb{N}$. So these semigroups don't have the Bergman Property.

This can be seen by taking U to be $B(0,2) \cap S$.

Theorem 3.1.3. *Let Ω be an infinite set and let $\{A, B, C\}$ be a partition of Ω into moieties of Ω. Then we have*

$$\text{Sym}(\Omega) = \text{Pstab}(A)\,\text{Pstab}(B)\,\text{Pstab}(A) \cup \text{Pstab}(B)\,\text{Pstab}(A)\,\text{Pstab}(B).$$

Proof. The following proof is based on the proof of Lemma 2.1 found in [3].
Let $f \in \text{Sym}(\Omega)$.
Case 1: $|\Omega| = |C\backslash((A)f^{-1})|$. The diagram below demonstrates the idea behind how to construct f as such a product. The colours demonstrating where points are as the permutations are applied, and the labelled regions saying where they originated.



As $|\Omega| = |C\backslash((A)f^{-1})|$, it follows that $|\Omega| = |C| = |(B \cup C)f \cap C| = |(B \cup C)f \cap (B \cup C)|$. Let $\{M_1, M_2\}$ be a partition of C into moieties and let $M_2' \subseteq M_2$ be such that $|M_2'| = |(C \cup B) \cap Af^{-1}|$. Let $b_1 : (C \cup B) \cap Af^{-1} \to M_2'$ and $b_2 : (C \cup B) \cap (C \cup B)f^{-1} \to (C \cup B)\backslash M_2'$ be bijections. Let $\phi_1 : \Omega \to \Omega$ be defined by

$$(x)\phi_1 = \left\{ \begin{array}{lr} x & x \in A \\ (x)b_1 & x \in (C \cup B) \cap Af^{-1} \\ (x)b_2 & x \in (C \cup B) \cap (C \cup B)f^{-1} \end{array} \right\}.$$

Let $b_3 : M_1 \cup (M_2 \backslash M_2') \cup (A \backslash Af^{-1}) \to C$ be a bijection (note this must exist as $|C| = |M_1| = |\Omega|$). Let $\phi_2 : \Omega \to \Omega$ be defined by

$$(x)\phi_2 = \left\{ \begin{array}{ll} x & x \in B \\ (x)b_3 & x \in M_1 \cup (M_2 \backslash M_2') \cup (A \backslash Af^{-1}) \\ (x)b_1^{-1}f & x \in M_2' \\ (x)f & x \in A \cap (Af^{-1}) \end{array} \right\}.$$

We have that $\phi_1 \in \operatorname{Pstab}(A)$ and $\phi_2 \in \operatorname{Pstab}(B)$. In addition $f^{-1}\phi_1\phi_2$ is a bijection which fixes A pointwise so $f^{-1}\phi_1\phi_2 \in \operatorname{Pstab}(A)$. So there exists $\phi_3 \in \operatorname{Pstab}(A)$ such that $f^{-1}\phi_1\phi_2 = \phi_3$. It therefore follows that $f = \phi_1\phi_2\phi_3^{-1} \in \operatorname{Pstab}(A)\operatorname{Pstab}(B)\operatorname{Pstab}(A)$ as required.
Case 2: If $|\Omega| = |C \backslash ((B)f^{-1})|$, we similarly conclude that $f \in \operatorname{Pstab}(B)\operatorname{Pstab}(A)\operatorname{Pstab}(B)$.
As $C = (C \backslash Af^{-1}) \cup (C \backslash Bf^{-1})$ and $|C| = |\Omega|$ we must be in one of these cases and thus the proof is complete. □

Definition 3.1.4 A semigroup S is said to be *quasi-bounded* if it satisfies the following:
Every function $\psi : S \to \mathbb{N}$ such that there is some constant $C_\psi \in \mathbb{N}$ satisfying

$$(st)\psi \leq (s)\psi + (t)\psi + C_\psi \quad \text{for all } s, t \in S$$

is bounded above.

Theorem 3.1.5. *Let S be a quasi-bounded semigroup. Then S also satisfies the semigroup Bergman property.*

Proof. Let $U \subseteq S$ be such that $\langle U \rangle_S = S$. Define $\psi : S \to \mathbb{N}$ by

$$(s)\psi = \min \left\{ n \in \mathbb{N} : s \in \bigcup_{i=1}^{n} U^i \right\}.$$

Let $C_\psi = 0$ and let $s, t \in S$. Observe that if $(s)\psi = l_1$ and $(t)\psi = l_2$, then we have that $s = u_{s_1}u_{s_2}\dots u_{s_{l_1}}$ and $t = u_{t_1}u_{t_2}\dots u_{t_{l_2}}$ for some $u_{s_1}, u_{s_2}\dots u_{s_{l_1}}, u_{t_1}, u_{t_2}\dots u_{t_{l_2}} \in U$.
Therefore $st = u_{s_1}u_{s_2}\dots u_{s_{l_1}}u_{t_1}u_{t_2}\dots u_{t_{l_2}} \in \bigcup_{i=1}^{l_1+l_2} U^i$ and therefore $(st)\psi \leq l_1 + l_2 = (s)\psi + (t)\psi + C_\psi$. As S is quasi-bounded we have that ψ is bounded by some $N \in \mathbb{N}$ and therefore $(s)\psi \leq N$ for all $s \in S$. This implies that $s \in \bigcup_{i=1}^{N} U^i$ for all $s \in S$ so we have $S = \bigcup_{i=1}^{N} U^i$. It follows that S satisfies the semigroup Bergman property. □

Theorem 3.1.6. *Let Ω be an infinite set. There is a sequence $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ and $N \in \mathbb{N}$, such that for all $S = (s_n)_{n\in\mathbb{N}} \subseteq \operatorname{Sym}(\Omega)$, we can find $G \subseteq \operatorname{Sym}(\Omega)$ such that for all $n \in \mathbb{N}$ we have $s_n \in \cup_{i=1}^{a_n} G^i$ and $|G| = N$.*

Proof. The following proof is based on the proof of Theorem 3.1 found in [3].
Choose $(a_n)_{n\in\mathbb{N}} = (36n + 6)_{n\in\mathbb{N}}, \quad N = 8$.
Let $(s_n)_{n\in\mathbb{N}}$ be a sequence in $\operatorname{Sym}(\Omega)$. Without loss of generality we assume that $\Omega = \mathbb{Z} \times \mathbb{Z} \times \Omega'$ where $|\Omega'| = |\Omega|$.
Let M be a moiety of Ω'.

$$\Omega_0 := \{0\} \times \{0\} \times \Omega', \qquad \Omega_0^+ := \{(0, 0, x) : x \in M\}, \qquad \Omega_0^- := \{(0, 0, x) : x \in M^c\}.$$

By Theorem 3.1.3, we can build a sequence $S' = (s_n')_{n\in\mathbb{N}} \subseteq \operatorname{Pstab}(\Omega_0^c) \cup \operatorname{Pstab}(\Omega_0^+)$ such that $s_n = s_{3n-2}'s_{3n-1}'s_{3n}'$ for all $n \in \mathbb{N}$.
Let $b_1 : \Omega_0^c \to \Omega_0^+$ be a bijection and let $\phi_1 : \Omega \to \Omega$, $\phi_2 : \Omega \to \Omega$ and $\phi_3 : \Omega \to \Omega$ be defined by:

$$(x)\phi_1 = \left\{ \begin{array}{ll} (x)b_1 & x \in \Omega_0^c \\ (x)b_1^{-1} & x \in \Omega_0^+ \\ x & x \in \Omega_0^- \end{array} \right\}, \qquad \begin{array}{l} ((a, b, c))\phi_2 = (a + 1, b, c), \\ ((a, b, c))\phi_3 = \left\{ \begin{array}{ll} (a, b + 1, c) & a = 0 \\ (a, b, c) & a \neq 0 \end{array} \right\}. \end{array}$$

Let the bijection $b_2 : \mathbb{Z} \to \mathbb{N}$ and s_i'' for $i \in \mathbb{Z}$ be defined by:

$$(i)b_2 = \left\{ \begin{array}{ll} 2i & i \geq 0 \\ 2|i| - 1 & i < 0 \end{array} \right\}, \qquad s_i'' = \left\{ \begin{array}{ll} s_{(i)b_2}' & s_{(i)b_2}' \in \operatorname{Pstab}(\Omega_0^c) \\ \phi_1 s_{(i)b_2}' \phi_1 & s_{(i)b_2}' \in S' \backslash \operatorname{Pstab}(\Omega_0^c) \end{array} \right\}.$$

Let $S'' := \{s_i'' : i \in \mathbb{Z}\}$. As S'' stabilises Ω_0^c pointwise for each s_i'' we can construct $\hat{s_i''} : \Omega' \to \Omega'$ such that $(0, 0, p)s_i'' = (0, 0, (p)\hat{s_i''})$ for all $p \in \Omega'$ so $\phi_4 : \Omega \to \Omega$ can be defined by

$$((a, b, c))\phi_4 = \left\{ \begin{array}{ll} (a, b, (c)\hat{s_a''}) & b \geq 0 \\ (a, b, c) & b < 0 \end{array} \right\}.$$

Let $(g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8) := (\phi_1, \phi_2, \phi_3, \phi_4, \phi_1^{-1}, \phi_2^{-1}, \phi_3^{-1}, \phi_4^{-1})$ and let $G := \{g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8\}$. Let $L(s)$ denote the minimum length of s as a product of elements of G. It suffices to show that $L(s_n) \leq a_n$ for all $n \in \mathbb{N}$.
Let $a, b, i \in \mathbb{Z}$ and $c \in \Omega'$. We have that:

$$((a,b,c))\phi_2^i\phi_4\phi_2^{-i} = ((a+i,b,c))\phi_4\phi_2^{-i} = \left\{ \begin{array}{ll} (a+i,b,(c)\hat{s''_{a+i}})\phi_2^{-i} & b \geq 0 \\ (a+i,b,c)\phi_2^{-i} & b < 0 \end{array} \right\} = \left\{ \begin{array}{ll} (a,b,(c)\hat{s''_{a+i}}) & b \geq 0 \\ (a,b,c) & b < 0 \end{array} \right\},$$

$$((0,b,c))\phi_3^{-1}\phi_2^i\phi_4^{-1}\phi_2^{-i}\phi_3 = ((i,b-1,c))\phi_4^{-1}\phi_2^{-i}\phi_3 = \left\{ \begin{array}{ll} (i,b-1,(c)\hat{s''^{-1}_i})\phi_2^{-i}\phi_3 & b \geq 1 \\ (i,b-1,c)\phi_2^{-i}\phi_3 & b < 1 \end{array} \right\} = \left\{ \begin{array}{ll} (0,b,(c)\hat{s''^{-1}_i}) & b \geq 1 \\ (0,b,c) & b < 1 \end{array} \right\}.$$

It follows that

$$(a,b,c)\phi_2^i\phi_4\phi_2^{-i}(\phi_3^{-1}\phi_2^i\phi_4^{-1}\phi_2^{-i}\phi_3) = \left\{ \begin{array}{lr} (a,b,c)\phi_2^i\phi_4\phi_2^{-i}(\phi_2^i\phi_4^{-1}\phi_2^{-i}\phi_3) & a \neq 0 \\ (a,b,c)\phi_3^{-1}\phi_2^i\phi_4^{-1}\phi_2^{-i}\phi_3 & a = 0 \text{ and } b < 0 \\ (a,b,(c)\hat{s''_{a+i}})\phi_3^{-1}\phi_2^i\phi_4^{-1}\phi_2^{-i}\phi_3 & a = 0 \text{ and } b > 0 \\ (a,b,(c)\hat{s''_{a+i}})\phi_3^{-1}\phi_2^i\phi_4^{-1}\phi_2^{-i}\phi_3 & a = b = 0 \end{array} \right\}$$

$$= \left\{ \begin{array}{lr} (a,b,c) & a \neq 0 \\ (a,b,c) & a = 0 \text{ and } b < 0 \\ (a,b,c) & a = 0 \text{ and } b > 0 \\ (a,b,(c)\hat{s''_{a+i}}) & a = b = 0 \end{array} \right\} = s''_i.$$

Thus for $i \geq 0$ we have $s''_i = g_2^i g_4 g_6^i g_7 g_2^i g_8 g_6^i g_3$ so $L(s''_i) \leq 4|i| + 4$. In addition for $i < 0$ we have $s''_i = g_6^{|i|} g_4 g_2^{|i|} g_7 g_6^{|i|} g_8 g_2^{|i|} g_3$ so $L(s''_i) \leq 4|i| + 4$.
For all $n \in \mathbb{N}$ we have that $s'_n = s''_{(n)b_2^{-1}}$ or $s'_n = \phi_1 s''_{(n)b_2^{-1}}\phi_1 = g_1 s''_{(n)b_2^{-1}} g_1$ so $L(s'_n) \leq L(s''_{(n)b_2^{-1}}) + 2$.
Therefore we have

$$\begin{aligned} L(s_n) &= L(s'_{3n-2}s'_{3n-1}s'_{3n}) \\ &\leq L(s'_{3n-2}) + L(s'_{3n-1}) + L(s'_{3n}) \\ &\leq L(s''_{(3n-2)b_2^{-1}}) + L(s''_{(3n-1)b_2^{-1}}) + L(s''_{(3n)b_2^{-1}}) + 6 \\ &\leq 4|(3n-2)b_2^{-1}| + 4|(3n-1)b_2^{-1}| + 4|(3n)b_2^{-1}| + 18 \\ &\leq 4(3n-2) + 4(3n-1) + 4(3n) + 18 \\ &\leq 36n + 6 = a_n. \end{aligned}$$

□

Theorem 3.1.7. *If Ω is an infinite set then* $\mathrm{Sym}(\Omega)$ *is quasi-bounded and satisfies the semigroup Bergman Property.*

Proof. The following proof is based on the proof of Lemma 2.4 in [4].
By Theorem 3.1.5 it suffices to show that $\mathrm{Sym}(\Omega)$ is quasi-bounded.
Let $(a_n)_{n\in\mathbb{N}}$ be as in Theorem 3.1.6.
We may assume a_n is strictly increasing as a_n can be replaced by $\max\{a_m + 1 : m \leq n\}$ and the required property holds.
Suppose for a contradiction that there exists $\psi : \mathrm{Sym}(\Omega) \to \mathbb{N}$ such that there exists C_ψ such that

$$(st)\psi \leq (s)\psi + (t)\psi + C_\psi \quad \text{for all } s, t \in \mathrm{Sym}(\Omega)$$

and ψ is unbounded.
As ψ is unbounded, for all $n \in \mathbb{N}$ there exists $s \in \mathrm{Sym}(\Omega)$ such that $(s)\psi > n$.
Therefore we can construct a sequence $(s_n)_{n\in\mathbb{N}}$ such that $(s_n)\psi > a_n^2$ for all $n \in \mathbb{N}$.
We now construct a set of generators $G = (g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8)$ for $(s_n)_{n\in\mathbb{N}}$ as done in Theorem 3.1.6.
Let $M := \max\{(g)\psi : g \in G\}$. Each s_n can be written as a product of length at most a_n in elements of G it therefore follows from induction that $(s_n)\psi \leq a_n C_\psi + a_n M$ for all $n \in \mathbb{N}$. For all sufficiently large n we have that $a_n > C_\psi + M$ and therefore

$$(s_n)\psi \leq a_n C_\psi + a_n M = a_n(C_\psi + M) < a_n^2 < (s_n)\psi.$$

So $(s_n)\psi < (s_n)\psi$. This is a contradiction. □

3.2 Cofinality and Strong Cofinality

In this section we will be exploring the cofinality and strong cofinality of infinite symmetric groups.

Definition 3.2.1 Let S be a semigroup. A *cofinal chain* of S, is defined to be a chain of strict subsemigroups $(S_i)_{i\in\kappa}$ of S indexed by the ordinals less than some cardinal κ such that

$$S = \bigcup_{i\in\kappa} S_i$$

and $S_i \subseteq S_j$ for $i \leq j$.

Definition 3.2.2 Let S be a semigroup. A *strong cofinal chain* of S, is defined to be a chain of strict subsets $(S_i)_{i\in\kappa}$ of S indexed by the ordinals less than some cardinal κ such that

$$S = \bigcup_{i\in\kappa} S_i$$

and $S_i \subseteq S_j$ for $i \leq j$ and for all $i \in \kappa$ there exists $j < \kappa$ such that $S_iS_i \subseteq S_j$.

Theorem 3.2.3. *If S is a non-finitely generated semigroup, then there exist cofinal and strong cofinal chains of S.*

Proof. The following proof is based on Note 3 of [5].
Let $S := \{t_i : i < |S|\}$ be an enumeration of S. Let $S_i := \langle\{t_j : j < i\}\rangle_S$ for $i < |S|$. We will show that $(S_i)_{i<|S|}$ is a cofinal chain.
We have that $S = \bigcup_{i<|S|} S_i$ and $S_i \subseteq S_j$ for $i \leq j$. To see that the S_i are strict subsemigroups observe that each S_i is generated by a set indexed by an ordinal $i < |S|$ and therefore is generated by a set of strictly smaller cardinality. If S is countable it follows that $S \neq S_i$ as S_i is finitely generated, and if S is uncountable it follows that $S \neq S_i$ as $|S_i| < |S|$. We therefore have that $(S_i)_{i<|S|}$ is a cofinal chain. As each S_i is a semigroup it follows that $S_iS_i \subseteq S_i$ and therefore $(S_i)_{i<|S|}$ is also a strong cofinal chain as required. □

Note that the validity of the following two definitions follows from the previous theorem together with the fact that the cardinals are a subclass of the ordinals and thus any class of cardinals has a least element.

Definition 3.2.4 Let S be a non-finitely generated semigroup. The *cofinality* of S, denoted $cf(S)$, is defined to be the smallest cardinal κ such that there exists a cofinal chain of S indexed by κ.

Definition 3.2.5 Let S be a non-finitely generated semigroup. The *strong cofinality* of S, denoted $scf(S)$, is defined to be the smallest cardinal κ such that there exists a strong cofinal chain of S indexed by κ.

Note that $\mathrm{Sym}(\Omega)$ is not finitely generated for an infinite Ω as it is uncountable by Theorem 1.3.7 and therefore we can assign it a cofinality and a strong cofinality. In addition by the proof of Theorem 3.2.3 we have

$$scf(\mathrm{Sym}(\Omega)) \leq cf(\mathrm{Sym}(\Omega)) \leq |\,\mathrm{Sym}(\Omega)| = 2^{|\Omega|}.$$

Theorem 3.2.6. *If Ω is an infinite set, then $cf(\mathrm{Sym}(\Omega)) > \aleph_0$.*

Proof. Suppose for a contradiction that $cf(\mathrm{Sym}(\Omega)) \leq \aleph_0$. Then there is a sequence of strict subsemigroups $(S_i)_{i\in\mathbb{N}}$ of $\mathrm{Sym}(\Omega)$ who's union is $\mathrm{Sym}(\Omega)$. Let $\psi : \mathrm{Sym}(\Omega) \to \mathbb{N}$ be defined by

$$(f)\psi = \min\{n \in \mathbb{N} : f \in S_n\}.$$

Let $s, t \in \mathrm{Sym}(\Omega)$. Without loss of generality we may assume that $S_{\psi(s)} \subseteq S_{\psi(t)}$, it follows that $st \in S_{\psi(t)}$. Letting $C_\psi = 0$ we have $(st)\psi \leq (t)\psi \leq (t)\psi + (s)\psi + C_\psi$.
By Theorem 3.1.7 it follows that $\mathrm{Sym}(\Omega)$ is quasi-bounded and so this function is bounded above by some natural number N. It follows then that for all $f \in \mathrm{Sym}(\Omega)$, we have $f \in S_N$ so $S_N \not< \mathrm{Sym}(\Omega)$ this is a contradiction. □

Theorem 3.2.7. *If Ω is an infinite set, then $scf(\mathrm{Sym}(\Omega)) > \aleph_0$.*

Proof. The following proof is based on the proof of proposition 2.2 in [4].
Suppose for a contradiction that $scf(\mathrm{Sym}(\Omega)) \leq \aleph_0$. Then there is a chain of strict subsets $(S_i)_{i\in\mathbb{N}}$ of $\mathrm{Sym}(\Omega)$ such that $\mathrm{Sym}(\Omega) = \bigcup_{i\in\mathbb{N}} S_i$, $S_i \subseteq S_j$ for $i \leq j$ and for all $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $S_iS_i \subseteq S_j$.
We have that $\mathrm{Sym}(\Omega) = \bigcup_{i\in\mathbb{N}}\langle S_i\rangle_S$ and $\langle S_i\rangle_S \subseteq \langle S_j\rangle_S$ for $i \leq j$. From Theorem 3.2.6 we have that $cf(\mathrm{Sym}(\Omega)) > \aleph_0$ and so we must have that $\langle S_j\rangle_S = \mathrm{Sym}(\Omega)$ for some $j \in \mathbb{N}$. By Theorem 3.1.7 $\mathrm{Sym}(\Omega)$ has the Bergman property so it follows that $\mathrm{Sym}(\Omega) = \bigcup_{k=1}^{n} S_j^k$ for some $n \in \mathbb{N}$.
To reach the desired contradiction it suffices to show that $\mathrm{Sym}(\Omega) = \bigcup_{k=1}^{n} S_j^k \subseteq S_N$ for some $N \in \mathbb{N}$.
We have that $S_jS_j \subseteq S_{j_2}$ for some $j_2 \in \mathbb{N}$. It follows that $S_jS_jS_j \subseteq S_{j_2}S_j \subseteq (S_{\max j,j_2})^2 \subseteq S_{j_3}$ for some $j_3 \in \mathbb{N}$. By induction for all $i \leq n$ we can construct j_i such that $S_j^i \subseteq S_{j_i}$. It therefore follows that $\cup_{k=1}^{n} S_j^k \subseteq S_{\max\{j_i : i \leq n\}}$ as required. □

3.3 Shuffling Infinite Planes

Throughout the next two sections we will without loss of generality consider Ω as $A \times A$ where A is an abelian group of infinite order (note that there are abelian groups of all infinite cardinalities, for example $\bigoplus_{i\in\kappa}\mathbb{Z}_2$ where κ is your cardinal). We will consider the operation on A using additive notation. Let A be indexed by $\{a_i : i \in |A|\}$ with $a_0 = id_A$. This indexing gives us a well ordering on the elements of A. We will use the functions π_1, π_2 to be the projection of a tuple onto its first and second coordinates respectively and if we have functions $S_i, S_{i+1} \dots S_k$ we will use the notation $S_{i\to k}$ to denote $S_iS_{i+1}\dots S_k$. In this section we try to write the elements of $\mathrm{Sym}(\Omega)$ as the product of 'slides'.

Definition 3.3.1 Let $f : A \to A$ be a function. Then a vertical slide $v_f : \Omega \to \Omega$ is defined by

$$(x, y)v_f = (x, y + (x)f).$$

Similarly a horizontal slide $h_f : \Omega \to \Omega$ is defined by

$$(x, y)h_f = (x + (y)f, y).$$

We will use the word slide to refer to either of these.

Definition 3.3.2 Let V be used to denote the group of all vertical slides of Ω, and similarly let H denote the group of all horizontal slides of Ω. Note that these groups are abelian as A is abelian.

We will start by showing any moiety can be mapped into the diagonal line $\{(x, y) \in A \times A : x = y\}$ and then show from this that we can construct any element of $\mathrm{Sym}(\Omega)$.

Definition 3.3.3 Let $x \in A$. Then the vertical and horizonal lines of x are defined respectively by:

$$v_x = \{(x, y) : y \in A\}, \qquad h_x = \{(y, x) : y \in A\}.$$

The word line will be used to describe any set of either of these types.

Definition 3.3.4 Let $L \subseteq \Omega$ be a line and let $S \subseteq \Omega$. We say that L is S-contained if we have that $L \subseteq S$, we say L is S-disjoint if $L \subseteq S^c$ and we say that L is S-sporadic if we have neither of these.

Theorem 3.3.5. *If M is a moiety of Ω then either there are $|\Omega| = |A|$ M-sporadic horizontal lines, or there are $|\Omega|$ M-sporadic vertical lines.*

Proof. The following proof is based on the proof of Lemma 1 in [13].
Suppose not, then we have $|\{x \in A : v_x \text{ is } M\text{-sporadic }\}| < |A|$ and $|\{x \in A : h_x \text{ is } M\text{-sporadic }\}| < |A|$ and thus also $|\{x \in A : v_x \text{ is } M\text{-contained or } M\text{-disjoint}\}| = |A|$.
Case 1: If $|\{x \in A : v_x \text{ is } M\text{-contained }\}| = |A|$ and $|\{x \in A : v_x \text{ is M-disjoint }\}| = |A|$ then it follows that for all $y \in A$ we have h_y is M-sporadic a contradiction.
Case 2: If $|\{x \in A : v_x \text{ is } M\text{-contained }\}| < |A|$ then it follows that we have $|\{x \in A : v_x \text{ is } M\text{-contained or } M\text{-sporadic }\}| < |A|$ and $|\{x \in A : v_x \text{ is } M\text{-disjoint }\}| = |A|$. It follows that $\{x \in A : h_x \text{ is } M\text{-contained }\} = \emptyset$. As $|\{x \in A : h_x \text{ is } M\text{-sporadic }\}| < |A|$ we have that $|\{x \in A : h_x \text{ is } M\text{-sporadic or } M\text{-contained }\}| < |A|$. However

$$M \subseteq \{x \in A : v_x \text{ is } M\text{-sporadic or } M\text{-contained }\} \times \{x \in A : h_x \text{ is } M\text{-sporadic or } M\text{-contained }\}.$$

Thus it follows that $|M| < |A \times A| = |A|$ as M is a moiety this is a contradiction.
Case 3: If $|\{x \in A : v_x \text{ is } M\text{-disjoint }\}| < |A|$ then it follows that we have $|\{x \in A : v_x \text{ is } M\text{-disjoint or } M\text{-sporadic }\}| < |A|$ and $|\{x \in A : v_x \text{ is } M\text{-contained }\}| = |A|$. It follows that $\{x \in A : h_x \text{ is } M\text{-disjoint }\} = \emptyset$. As $|\{x \in A : h_x \text{ is } M\text{-sporadic }\}| < |A|$ we have that $|\{x \in A : h_x \text{ is } M\text{-sporadic or } M\text{-disjoint }\}| < |A|$. However

$$M^c \subseteq \{x \in A : v_x \text{ is } M\text{-sporadic or } M\text{-disjoint }\} \times \{x \in A : h_x \text{ is } M\text{-sporadic or } M\text{-disjoint }\}.$$

Thus it follows that $|M^c| < |A \times A| = |A|$ as M is a moiety this is a contradiction. □

Theorem 3.3.6. *If M is a moiety, then there exists a slide S such that either all horizontal lines are $(M)S$-sporadic, or all vertical lines are $(M)S$-sporadic.*

Proof. The following proof is based on the proof of Lemma 2 in [13].
By Theorem 3.3.5 we may assume without loss of generality that there are $|A|$ M-sporadic horizontal lines.
The idea is that there are 'lots' of M-sporadic horizontal lines, so for each vertical line we can slide 2 M-sporadic horizontal lines in such a way that they each put a specific point into our vertical line. One of which comes from M and one of which doesn't.

Let $\{M_1, M_2\}$ be a partition of $\{x \in A : h_x \text{ is } M\text{-sporadic}\}$ into moieties . Let $\phi_1 : M_1 \to A$ and $\phi_2 : M_2 \to A$ be bijections. Let $\phi_3 : \{x \in A : h_x \text{ is } M\text{-sporadic}\} \to A$ and $\phi_4 : \{x \in A : h_x \text{ is } M\text{-sporadic}\} \to A$ be such that for all $x \in A$ such that h_x is M-sporadic we have $((x)\phi_3, x) \in M$ and $((x)\phi_4, x) \notin M$.
Let $f : A \to A$ be defined by

$$(a)f = \left\{ \begin{array}{ll} -(a)\phi_3 + (a)\phi_1 & a \in M_1 \\ -(a)\phi_4 + (a)\phi_2 & a \in M_2 \\ a & \text{otherwise} \end{array} \right\}.$$

Let $S := h_f$. For all $x \in A$ we have

$$(x,(x)\phi_1^{-1}) = ((x)\phi_1^{-1}\phi_3 - (x)\phi_1^{-1}\phi_3 + (x)\phi_1^{-1}\phi_1, (x)\phi_1^{-1}) = ((x)\phi_1^{-1}\phi_3 + (x)\phi_1^{-1}f, (x)\phi_1^{-1}) = ((x)\phi_1^{-1}\phi_3, (x)\phi_1^{-1})h_f \in (M)S,$$

$$(x,(x)\phi_2^{-1}) = ((x)\phi_2^{-1}\phi_4 - (x)\phi_2^{-1}\phi_4 + (x)\phi_2^{-1}\phi_2, (x)\phi_2^{-1}) = ((x)\phi_2^{-1}\phi_4 + (x)\phi_2^{-1}f, (x)\phi_2^{-1}) = ((x)\phi_2^{-1}\phi_4, (x)\phi_2^{-1})h_f \notin (M)S$$

and thus v_x is $(M)S$-sporadic as required. □

For the rest of the section on shuffling infinite planes, it is required that the reader is familiar with the principals of transfinite recursion and ordinal multiplication. However the remaining sections after this one will not require this, and will also not require the remaining results of this section.

Definition 3.3.7 Let P be the set of 4-tuples (x, X, Y_0, Y_1) where $x \in A$, X is an initial segment of A, $Y_0, Y_1 \subseteq A$, $|Y_0|, |Y_1| \leq 2$ and X, Y_0, Y_1 are pairwise disjoint.

Theorem 3.3.8. *The set P can be indexed by an ordinal α as $\{p_i = (x_i, X_i, Y_{0,i}, Y_{1,i}) : i \in \alpha\}$, such that we have $\{p_i \in P : \{x_i\} \cup X_i \cup Y_{0,i} \cup Y_{1,i} \subseteq \{a_j : a_j < a_M\}\}$ is bounded above (in the well order given by the indexing of P) for all $a_M \in A$. In addition there exist $t_i \in A$ such that $\{t_i + X_i \cup Y_{0,i} \cup Y_{1,i} : i \in \alpha\}$ are pairwise disjoint.*

Proof. This proof is based on the proof of Lemma 3 in [13].
By the axiom of choice let $c_p : \mathcal{P}(P)\backslash\{\emptyset\} \to P$ and $c_a : \mathcal{P}(A)\backslash\{\emptyset\} \to A$ be choice functions.
By transfinite recursion we define the following for $i \in 2^{|\Omega|}$ (until P has been indexed):

1. $A_i := \bigcup_{j<i}(\{x_j\} \cup X_j \cup Y_{0,j} \cup Y_{1,j}\}),$
2. $p_i := c_p(\{(x, X, Y_0, Y_1) \in P : \{x\} \cup X \cup Y_0 \cup Y_1 \subseteq A_i\}\backslash\{p_j : j < i\})$ unless this set is empty in which case $p_i := (\min(A\backslash\bigcup_{j<i} A_j), \emptyset, \emptyset, \emptyset)$ (if this set is empty then we are done),
3. $t_i := c_a(A\backslash\bigcup_{j<i}(t_j + X_j \cup Y_{0,j} \cup Y_{1,j} - X_i \cup Y_{0,i} \cup Y_{1,i}))$.

Note that $A\backslash\cup_{j<i}(t_j + X_j \cup Y_{0,j} \cup Y_{1,j} - X_i \cup Y_{0,i} \cup Y_{1,i})$ is non-empty as we are removing less than $|A|$ points from A. Note that by construction we have $\{p_i \in P : \{x_i\} \cup X_i \cup Y_{0,i} \cup Y_{1,i} \subseteq \{a_j : j < M\}\}$ is bounded above by $(a_{M+2}, \emptyset, \emptyset, \emptyset)$, and if $\{t_i + X_i \cup Y_{0,i} \cup Y_{1,i} : i \in |\Omega|\}$ were not pairwise disjoint then we would have $(t_i + X_i \cup Y_{0,i} \cup Y_{1,i}) \cap (t_j + X_j \cup Y_{0,j} \cup Y_{1,j}) \neq \emptyset$ for some $i > j$ and therefore $t_i \in (t_j + X_j \cup Y_{0,j} \cup Y_{1,j} - X_i \cup Y_{0,i} \cup Y_{1,i})$ contradicting its definition. Finally $|\Omega| \leq |P| \leq |\Omega| \times |\Omega| \times 2 \times 2 = |\Omega|$ so $|\Omega| = |P|$ and so this process must terminate at some ordinal $\alpha < 2^{|\Omega|}$. □

Theorem 3.3.9. *Let M be a moiety of Ω. Then there exist slides S_1, S_2 such that for all $p_i \in P$ we have one of the following:*

1. $\{(x_i, t_i + b) : b \in Y_{1,i}\} \subseteq (M)S_{1\to 2}$ *and* $\{(x_i, t_i + c) : c \in X_i \cup Y_{0,i}\} \subseteq ((M)S_{1\to 2})^c$,
2. $\{(t_i + b, x_i) : b \in Y_{1,i}\} \subseteq (M)S_{1\to 2}$ *and* $\{(t_i + c, x_i) : c \in X_i \cup Y_{0,i}\} \subseteq ((M)S_{1\to 2})^c$.

Proof. The following proof is based on the proof of Lemma 3 in [13].
Let S_1 be as in Theorem 3.3.6 and suppose that all horizontal lines are $(M)S_1$-sporadic. By Theorem 3.3.8 we have $\{t_i + X_i \cup Y_{0,i} \cup Y_{1,i} : i \in \alpha\}$ are pairwise disjoint. Let $\phi_1 : A \to A$ and $\phi_2 : A \to A$ be such that $((x)\phi_1, x) \in (M)S_1$ and $((x)\phi_2, x) \notin (M)S_1$. Let $f : A \to A$ be defined by

$$(a)f = \left\{ \begin{array}{ll} -(a)\phi_1 + x_i & a \in (t_i + Y_{1,i}) \\ -(a)\phi_2 + x_i & a \in (t_i + X_i \cup Y_{0,i}) \\ a & \text{otherwise} \end{array} \right\}.$$

Let $S_2 := h_f$. For all $i \in \alpha$, $b \in Y_{1,i}$ and $c \in X_i \cup Y_{0,i}$ we have

$$(x_i, t_i + b) = ((t_i + b)\phi_1 - (t_i + b)\phi_1 + x_i, t_i + b) = ((t_i + b)\phi_1, t_i + b)h_f \in (M)S_{1\to 2},$$

$$(x_i, t_i + c) = ((t_i + c)\phi_2 - (t_i + c)\phi_2 + x_i, t_i + c) = ((t_i + c)\phi_2, t_i + c)h_f \notin (M)S_{1\to 2}.$$

Thus the first condition is satisfied.
If instead all vertical lines are $(M)S_1$-sporadic we can conclude by symmetry that condition 2 can be satisfied by a symmetric choice of S_2. □

Theorem 3.3.10. *Let M be a moiety of Ω. If Ω is countable then there exist slides S_1, S_2, S_3, S_4, S_5 such that $(M)S_{1\to5} \subseteq \{(x,y) \in A \times A : x = y\}$.*

Proof. The following proof is based on the proof of Lemma 4 in [13].
Let S_1, S_2 be as in Theorem 3.3.9. Without loss of generality for all $p_i \in P$ we have $\{(x_i, t_i + b) : b \in Y_{1,i}\} \subseteq (M)S_{1\to2}$ and $\{(x_i, t_i + c) : c \in X_i \cup Y_{0,i}\} \subseteq ((M)S_{1\to2})^c$. We proceed by induction. Let $k_0 = 0, X_0 = Y_0 = Z_0 = \{a_0\}, b_0 = c_0 = a_0$.

1. Let $X_{n+1} := \{a_i \in A : i \le k_n \text{ and } (a_i, a_{n+1} + c_i) \in (M)S_{1\to2}\}$.

2. Let $k_{n+1} > k_n$ be such that $(X_{n+1} + \min(A \setminus \bigcup_{i=0}^n Z_i) - a_{k_{n+1}}) \cap (\bigcup_{i=0}^n Z_i) = \emptyset$. This must exist as $|(X_{n+1} + \min(A \setminus \bigcup_{i=0}^n Z_i)) - (\bigcup_{i=0}^n Z_i)|$ is finite and therefore its complement must contain elements greater than a_{k_n}.

3. Let $b_{n+1} := \min(A \setminus \bigcup_{i=0}^n Z_i) - a_{k_{n+1}}$. Note this means that $(X_{n+1} + b_{n+1}) \cap (\bigcup_{i=0}^n Z_i) = \emptyset$ by the definition of k_{n+1}.

4. Let c_i for $k_n < i \le k_{n+1}$ be defined such that: $(a_{k_{n+1}}, a_{n+1} + c_{k_{n+1}}) \in (M)S_{1\to2}$ and for all other $k_n < i \le k_{n+1}$ and $j \le n+1$ we have $(a_i, a_j + c_i) \notin (M)S_{1\to2}$. This can be done by the definition of S_1, S_2 as $(a_{k_{n+1}}, \{a_j : j < n+1\}, \emptyset, \{a_{n+1}\}) \in P$ and $(a_i, \{a_j : j < n+1\}, \emptyset, \emptyset) \in P$ for $k_n < i < k_{n+1}$.

5. Let $Y_{n+1} := X_{n+1} \cup \{a_{k_{n+1}}\}$.

6. Let $Z_{n+1} := Y_{n+1} + b_{n+1}$. Note that Z_{n+1} is disjoint from $\bigcup_{i=0}^n Z_i$ (by 3).

As each Z_n contains $\min(A \setminus \cup_{i=0}^{n-1} Z_i)$ (by 3,5,6) we have that $\bigcup_{n\in\mathbb{N}} Z_n = A$. In addition (by 6) we have that the Z_n are disjoint. So we have the Z_n partition A.
Let $S_3 := v_{f_3}$ where f_3 is defined by $(a_i)f_3 = -c_i$. Let $S_4 := h_{f_4}$ where f_4 is defined by $(a_i)f_4 = b_i$. Let $S_5 := v_{f_5}$ where f_5 is defined by $(a_i)f_5 = a_i - a_{n_i}$ (where Z_{n_i} contains a_i).
We have that $Y_n = \{a_i \in A : (a_i, a_n) \in MS_{1\to3}\}$ as if $a_i \in Y_n$ we have that $(a_i, a_n + c_i) \in MS_{1\to2}$ (by 1,4,5) and if $a_i \notin Y_n$ we have that $(a_i, a_n + c_i) \notin MS_{1\to2}$ (by 1,5 if $i \le k_n$ and by 4,5 if $i > k_n$).
We therefore have (by 6) that $Z_n = \{a_i \in A : (a_i - b_n, a_n) \in MS_{1\to3}\} = \{a_i \in A : (a_i, a_n) \in MS_{1\to4}\}$.
So we have that if $(a_i, a_n) \in MS_{1\to4}$ then $a_i \in Z_n$ and thus $(a_i, a_n)S_5 = (a_i, a_n - a_n + a_i) = (a_i, a_i) \in \{(x,y) \in A \times A : x = y\}$ and thus $MS_{1\to5} \subseteq \{(x,y) \in A \times A : x = y\}$. □

Theorem 3.3.11. *Let M be a moiety of Ω. If Ω is uncountable then there exist slides S_1, S_2, S_3, S_4, S_5 such that $(M)S_{1\to5} \subseteq \{(x,y) \in A \times A : x = y\}$.*

Proof. The following proof is based on the proof of Lemma 4 in [13].
Let S_1, S_2 be as in Theorem 3.3.9. Without loss of generality for all $p_i \in P$ we have $\{(x_i, t_i + b) : b \in Y_{1,i}\} \subseteq (M)S_{1\to2}$ and $\{(x_i, t_i + c) : c \in X_i \cup Y_{0,i}\} \subseteq ((M)S_{1\to2})^c$. We first re-index $A \setminus \{id_A\} = \{a_i : i \in |A|\}$ and we define $(A_i)_{i\in|A|}$ to be a cofinal chain for A such that $A_0 = \{id_A\}$ and for all $i \in |A|$ we have A_i is a group, $a_{2i}, a_{2i+1} \in A_{i+1}$ and $[A_{i+1} : A_i] \ge 4$ (where $[G_1 : G_2]$ denotes the index of G_2 as a subgroup of G_1). This can be done by transfinite recursion as follows:

1. if $i = 0$ then let $A_i := \{id_A\}$,

2. if i is a successor ordinal let $A_i := \langle\{a_j : j \le \min\{k \ge 2i+1 : [\langle\{a_\alpha : \alpha \le k\}\rangle_G : A_i] \ge 4\}\}\rangle_G$,

3. if i is a limit ordinal then $A_i := \bigcup_{j<i} A_j$.

For $i \in |A|$ we define c_i, d_i, m_i by transfinite recursion.

1. Let $c_i \in A_{i+1} \setminus A_i$,

2. Let $d_i \in A_{i+1} \setminus ((A_i + c_i) \cup (A_i + c_i^{-1}) \cup A_i)$. Note that we can do this as $[A_{i+1} : A_i] \ge 4$. It follows from the definition of d_i that $id_A, c_i, d_i, c_i + d_i$ are in different cosets of A_i in A_{i+1},

3. For $a_j \in A_{i+1} \setminus A_i$ by the definition of S_1, S_2 we can define m_j to be such that:
   (a) $(a_j, a_k - m_j) \notin (M)S_{1\to2}$ for $k < 2i$ and $(a_j, -m_j) \notin (M)S_{1\to2}$,
   (b) If $a_j \in (A_i - c_i)$ then $(a_j, a_{2i} - m_j) \notin (M)S_{1\to2}$
   (c) If $a_j = a_k - d_i - c_i$ for some $a_k \in A_i$ and $(a_k, a_{2i+1} - m_k) \notin (M)S_{1\to2}$ then $(a_j, a_{2i} - m_j) \in (M)S_{1\to2}$,
   (d) If $a_j = a_k - d_i - c_i$ for some $a_k \in A_i$ and $(a_k, a_{2i+1} - m_k) \in (M)S_{1\to2}$ then $(a_j, a_{2i} - m_j) \notin (M)S_{1\to2}$,
   (e) If $a_j \in A_{i+1} \setminus (A_i \cup (A_i - c_i) \cup (A_i - (d_i + c_i)))$ then $(a_j, a_{2i} - m_j) \in (M)S_{1\to2}$,
   (f) If $a_j = a_k + c_i + d_i$ for some $a_k \in A_i$ and $(a_k, a_{2i} - m_k) \notin (M)S_{1\to2}$ then $(a_j, a_{2i+1} - m_j) \in (M)S_{1\to2}$,
   (g) If $a_j = a_k + c_i + d_i$ for some $a_k \in A_i$ and $(a_k, a_{2i} - m_k) \in (M)S_{1\to2}$ then $(a_j, a_{2i+1} - m_j) \notin (M)S_{1\to2}$,
   (h) If $a_j \in A_{i+1} \setminus (A_i \cup (A_i + (c_i + d_i)))$ then $(a_j, a_{2i+1} - m_j) \notin (M)S_{1\to2}$.

Let $S_3 := v_{f_3}$ where f_3 is defined by $(a_i)f_3 = m_i$ and $(id_A)f_3$ is such that $(id_A, -(id_A)f_3) \in (M)S_{1\to 2}$. Let $S_4 := h_{f_4}$ where f_4 is defined by $(a_{2i})f_4 = c_i$, $(a_{2i+i})f_4 = -d_i$ and $(id_A)f_4 = id_A$. Note that for all non-identity elements $x \in A$ there is a unique $i \in |A|$ such that $x \in A_{i+1}\backslash A_i$. We will now show that for all $x \in A$ there is at most one point in $MS_{1\to 4} \cap v_x$. For $i \in |A|$ we have (by 3b and 3h) that:

$$(-c_i, a_{2i} - (-c_i)f_3) \notin (M)S_{1\to 2} \implies (-c_i, a_{2i}) \notin (M)S_{1\to 3} \implies (id_A, a_{2i}) \notin (M)S_{1\to 4},$$
$$(d_i, a_{2i+1} - (d_i)f_3) \notin (M)S_{1\to 2} \implies (d_i, a_{2i+1}) \notin (M)S_{1\to 3} \implies (id_A, a_{2i+1}) \notin (M)S_{1\to 4}.$$

Let $a_j \in A_{i+1}\backslash A_i$.
For $k < i$ we have $a_j - c_k \in A_{i+1}\backslash A_i$ and $a_j + d_k \in A_{i+1}\backslash A_i$ therefore (by 3a) we have $(a_j - c_k, a_{2k} - (a_j - c_k)f_3) \notin (M)S_{1\to 2}$ and $(a_j + d_k, a_{2k+1} - (a_j + d_k)f_3) \notin (M)S_{1\to 2}$. For $k > i$ we also have these two conditions (by 3b and 3h).
Therefore for $i \neq k$ we have:

$$(a_j - c_k, a_{2k} - (a_j - c_k)f_3) \notin (M)S_{1\to 2} \implies (a_j - c_k, a_{2k}) \notin (M)S_{1\to 3} \implies (a_j, a_{2k}) \notin (M)S_{1\to 4} \implies (a_j, a_j + a_{2k} - a_{2i}),$$
$$(a_j + d_k, a_{2k+1} - (a_j + d_k)f_3) \notin (M)S_{1\to 2} \implies (a_j + d_k, a_{2k+1}) \notin (M)S_{1\to 3} \implies (a_j, a_{2k+1}) \notin (M)S_{1\to 4}.$$

If $a_j = a_k - d_i$ for some $a_k \in A_i$ then (by 3d) we have one of:

$$(a_k, a_{2i+1} - m_k) \notin (M)S_{1\to 2} \implies (a_k, a_{2i+1}) \notin (M)S_{1\to 3} \implies (a_j, a_{2i+1}) \notin (M)S_{1\to S_4},$$
$$(a_j - c_i, a_{2i} - (a_j - c_i)f_3) \notin (M)S_{1\to 2} \implies (a_j - c_i, a_{2i}) \notin MS_{1\to 3} \implies (a_j, a_{2i}) \notin (M)S_{1\to 4}.$$

If $a_j = a_k + c_i$ for some $a_k \in A_i$. then (by 3g) we have one of:

$$(a_k, a_{2i} - m_k) \notin MS_{1\to 2} \implies (a_k, a_{2i}) \notin MS_{1\to 3} \implies (a_j, a_{2i}) \notin MS_{1\to 4},$$
$$(a_j + d_i, a_{2i+1} - (a_j + d_i)f_3) \notin MS_{1\to 2} \implies (a_j + d_i, a_{2i+1}) \notin MS_{1\to 3} \implies (a_j, a_{2i+1}) \notin MS_{1\to 4}.$$

Otherwise by (3h) we have

$$(a_j + d_i, a_{2i+1} - (a_j + d_i)f_3) \notin MS_{1\to 2} \implies (a_j + d_i, a_{2i+1}) \notin MS_{1\to 3} \implies (a_j, a_{2i+1}) \notin MS_{1\to 4}.$$

We therefore have that $v_{id_A} \cap (M)S_{1\to 5}$ contains at most (id_A, id_A) and for all a_j we have at most one of $(a_j, a_{2i}), (a_j, a_{2i+1})$ in $v_{a_j} \cap (M)S_{1\to 4}$ and no other points. Thus we can construct a vertical slide S_5 such that $MS_{1\to 5} \subseteq \{(x, y) \in A \times A : x = y\}$. □

Theorem 3.3.12. *If M is a moiety of Ω, then there exist slides S_1, S_2, S_3, S_4, S_5 such that $(M)S_{1\to 5} \subseteq \{(x, y) \in A \times A : x = y\}$.*

Proof. Take the previous two theorems together. □

Now that we can easily permute moieties by first moving them into a diagonal, permuting the diagonal, and moving them back again.

Theorem 3.3.13. *Let M be a moiety and let $p \in \mathrm{Sym}_\Omega(M)$ then there exist 11 slides $S_1, S_2 \ldots S_{11}$ such that $S_{1\to 11}|_M = p|_M$.*

Proof. The following proof is based on the proof of Claim 11 in [12].
Let S_1, S_2, S_3, S_4, S_5 be as in Theorem 3.3.12 and for $6 < n < 12$ let $S_n := S_{12-n}^{-1}$. Without loss of generality assume that S_5 is a vertical slide. Let $I : M \to A$ be defined by $(x)I = (x)S_{1\to 5}\pi_1 = (x)S_{1\to 5}\pi_2$.
Let $f_1 : A \to A$ and $f_2 : A \to A$ be defined by:

$$(a)f_1 = \left\{ \begin{array}{ll} (a)I^{-1}pI - a & a \in \mathrm{img}(I) \\ a_0 & \text{otherwise} \end{array} \right\}, \quad (a)f_2 = \left\{ \begin{array}{ll} a - (a)I^{-1}p^{-1}I & a \in \mathrm{img}(I) \\ a_0 & \text{otherwise} \end{array} \right\}.$$

Let $S_5' := v_{f_1}$ and $S_6 := h_{f_2}$. We now have for $x \in M$

$$\begin{aligned}
(x)S_{1\to 5}S_5'S_{6\to 11} &= ((x)I, (x)I)S_5'S_{6\to 11} \\
&= ((x)I, (x)I + (x)II^{-1}pI - (x)I)S_{6\to 11} \\
&= ((x)I, (x)pI)S_{6\to 11} \\
&= ((x)I + (x)pI - (x)pII^{-1}p^{-1}I, (x)pI)S_{7\to 11} \\
&= ((x)I + (x)pI - (x)I, (x)pI)S_{7\to 11} \\
&= ((x)pI, (x)pI)S_{7\to 11} \\
&= (x)p.
\end{aligned}$$

Thus $S_{1\to 5}S_5'S_{6\to 11}|_M = p|_M$ and as S_5 and S_5' are both vertical slides it follows that S_5S_5' is also a single vertical slide and thus we have the required result. □

Theorem 3.3.14. *Let M be a moiety of Ω and let $p \in \mathrm{Sym}_\Omega(M)$. Then there are slides $S_1, S_2 \dots S_{44}$ such that $p = S_{1\to 44}$.*

Proof. The following proof is based on the proof of Claim 12 in [12].
Let $\{M_1, M_2\}$ be a partition of M^c into moieties. By Theorem 2.3.7 let $p_1, p_2 \in \mathrm{Sym}_\Omega(M)$ be such that $p_1^{-1}p_2^{-1}p_1p_2 = p$. By Theorem 3.3.13 let $S_1, S_2 \dots S_{22}$ be such that $S_{1\to 11}|_{M\cup M_1} = p_1^{-1}|_{M\cup M_1}$ and $S_{12\to 22}|_{M\cup M_2} = p_2^{-1}|_{M\cup M_2}$. Let $S_n := S_{34-n}^{-1}$ for $23 \le n \le 33$ and $S_n = S_{56-n}^{-1}$ for $34 \le n \le 44$.
Let $(x,y) \in \Omega$.

$$\begin{aligned}
(x,y)S_{1\to 44} &= \left\{ \begin{array}{ll} (x,y)p_1^{-1}S_{12\to 44} & (x,y) \in M \\ (x,y)S_{12\to 44} & (x,y) \in M_1 \\ ((x,y)S_{1\to 11})S_{12\to 44} & (x,y) \in M_2 \end{array} \right\} \\
&= \left\{ \begin{array}{ll} (x,y)p_1^{-1}p_2^{-1}S_{23\to 44} & (x,y) \in M \\ ((x,y)S_{12\to 22})S_{23\to 44} & (x,y) \in M_1 \\ ((x,y)S_{1\to 11})S_{23\to 44} & (x,y) \in M_2 \end{array} \right\} \\
&= \left\{ \begin{array}{ll} (x,y)p_1^{-1}p_2^{-1}p_1S_{34\to 44} & (x,y) \in M \\ ((x,y)S_{12\to 22})S_{34\to 44} & (x,y) \in M_1 \\ (x,y)S_{34\to 44} & (x,y) \in M_2 \end{array} \right\} \\
&= \left\{ \begin{array}{ll} (x,y)p_1^{-1}p_2^{-1}p_1p_2 & (x,y) \in M \\ (x,y) & (x,y) \in M_1 \\ (x,y) & (x,y) \in M_2 \end{array} \right\} \\
&= (x,y)p.
\end{aligned}$$

□

The following theorem gives a slight alteration to the idea given in [12]. This slightly improves the number of slides needed to generate the entire symmetric group (from 81 to 55).

Theorem 3.3.15. *Let $p \in \mathrm{Sym}(\Omega)$, then there exist 55 slides $S_1, S_2 \dots S_{55}$ such that $p = S_{1\to 55}$.*

Proof. Let P be the partition of Ω into disjoint cycles of p.
Case 1: If we have that $|P| = |\Omega|$ then let M_p be a moiety of P and let $M := \cup M_p$. We have that $p = p|_M p|_{M^c}$. By Theorem 3.3.13 let $S_1, S_2 \dots S_{11}$ be such that $p|_M = S_{1\to 11}|_M$. By Theorem 3.3.14 we can find $S_{12,13,14\dots 55}$ such that $(S_{1\to 11})^{-1}(p|_M)(p|_{M^c}) = S_{12\to 55}$ as this product fixes the moiety M pointwise. We therefore have that

$$S_{1\to 55} = (S_{1\to 11})S_{12\to 55} = (S_{1\to 11})(S_{1\to 11})^{-1}(p|_M)(p|_{M^c}) = (p|_M)(p|_{M^c}) = p.$$

Case 2: If we have that $|P| < |\Omega|$ then as each cycle is countable it follows that Ω is countable. As $|\Omega| = \aleph_0$ we have that p has finitely many cycles and thus has at least one infinite cycle $C = (\dots c_{-1}, c_0, c_1 \dots)$. Let $p_1 : \Omega \to \Omega$ be defined by

$$(x)p_1 = \left\{ \begin{array}{ll} c_{i+1} & x = c_i \text{ for some } i = 0 \bmod 3 \\ c_{i-1} & x = c_i \text{ for some } i = 1 \bmod 3 \\ x & \text{otherwise} \end{array} \right\}.$$

Let $M_1 := \{c_i : i \neq 2 \bmod 3\}$ and $M_2 := \{c_i : i = 1 \bmod 3\}$. By Theorem 3.3.13 let $S_1, S_2 \dots S_{11}$ be such that $p_1|_{M_1} = S_{1\to 11}|_{M_1}$. By Theorem 3.3.14 we can find $S_{12\to 55}$ such that $(S_{1\to 11})^{-1}p = S_{12\to 55}$ as this product fixes the moiety M_2 pointwise. We therefore have that

$$S_{1\to 55} = (S_{1\to 11})S_{12\to 55} = (S_{1\to 11})(S_{1\to 11})^{-1}p = p.$$

□

Corollary 3.3.16. *The group $\mathrm{Sym}(\Omega)$ is equal to $(HV)^{28}$.*

Proof. Let $p \in \mathrm{Sym}(\Omega)$, we can now write p as $S_{1\to 55}$ this is an alternating product of elements of H and V so if $S_1 \in H$ then $S_{1\to 55} \in (HV)^{27}H \subseteq (HV)^{28}$ and if $S_1 \in V$ then $S_{1\to 55} \in V(HV)^{27} \subseteq (HV)^{28}$ □

3.4 Products of Abelian Groups

In the previous section on shuffling the plane, the works of Miklos Abert, Tamas Keleti and Peter Komjath gave us a means for expressing any element of an infinite symmetric group as a product of 'slides'. This gives us a way of writing any infinite symmetric group as a product of 56 abelian groups. In [14] Akos Seress gives a way to write any infinite symmetric group as a product of 14 abelian groups. In this section we will further decrease this bound to 10.

Definition 3.4.1 For $t \in A$, we call a set $D \subseteq \{(a, t+a) : a \in A\}$ a t-diagonal segment.

Observe that if $a \neq b$ then all a-diagonal segments are disjoint from all b-diagonal segments.

Theorem 3.4.2. *Let $t \in A$. If D is a t-diagonal segment, then HV acts fully on D.*

Proof. The following proof is based on the proof of Lemma 3 in [14]:
Let $g \in \mathrm{Sym}(D)$ and let $f_1 : A \to A$ and $f_2 : A \to A$ be defined by:

$$(a)f_1 = (a-t, a)g\pi_1 - a + t, \qquad (a)f_2 = a + t - (a, a+t)g^{-1}\pi_2.$$

Let $(p, p+t) \in D$, we have that

$$\begin{aligned}
(p, p+t)h_{f_1}v_{f_2} &= (p + (p+t-t, p+t)g\pi_1 - (p+t) + t, p+t)v_{f_2} \\
&= ((p, p+t)g\pi_1, p+t)v_{f_2} \\
&= ((p, p+t)g\pi_1, p+t + (p, p+t)g\pi_1 + t - ((p, p+t)g\pi_1, (p, p+t)g\pi_1 + t)g^{-1}\pi_2) \\
&= ((p, p+t)g\pi_1, p+t + (p, p+t)g\pi_1 + t - ((p, p+t)g\pi_1, (p, p+t)g\pi_2)g^{-1}\pi_2) \\
&= ((p, p+t)g\pi_1, p+t + (p, p+t)g\pi_1 + t - (p, p+t)gg^{-1}\pi_2) \\
&= ((p, p+t)g\pi_1, p+t + (p, p+t)g\pi_1 + t - (p+t)) \\
&= ((p, p+t)g\pi_1, (p, p+t)g\pi_1 + t) \\
&= ((p, p+t)g\pi_1, (p, p+t)g\pi_2) \\
&= (p, p+t)g.
\end{aligned}$$

So we have that $h_{f_1}v_{f_2} \in HV$ satisfies that $(h_{f_1}v_{f_2})|_D = g$ as required. □

Theorem 3.4.3. *If M is a moiety of Ω and $t \in A$, then there exists $h_1v_1h_2 \in HVH$ such that $Mh_1v_1h_2 \cap D = \emptyset$ for all t-diagonal segments D.*

Proof. By Theorem 3.3.6 we have that there is a slide $S \in H \cup V$ such that either all horizontal lines are $(M)S$-sporadic or all vertical lines are $(M)S$-sporadic.
Case 1: If $S \in H$, and all vertical lines are $(M)S$-sporadic, then for all $x \in A$ choose $p_x \in v_x \backslash (M)S$. Let $f : A \to A$ be defined by $(a)f = a + t - p_a$. It follows that for all $(x, y) \in M$ we have $(x, y)Sv_f \notin D$ and thus we can choose $h_1 = S, v_1 = v_f$ and $h_2 = id_A$.
Case 2: If $S \in H$, and all horizontal lines are $(M)S$-sporadic, then we can make a similar argument by using only an element of H (viewed as a product of two elements of H)
Case 3: If $S \in V$, and all horizontal lines are $(M)S$-sporadic, then we can make a similar argument by letting $h_1 := id_A$ and $v_1 = S$.
Case 4: If $S \in V$, and all vertical lines are $(M)S$-sporadic, then we can make a similar argument by using only an element of V (viewed as a product of two elements of V) □

Theorem 3.4.4. *If M be a moiety of Ω, then there exist abelian groups H_M, V_M such that H_MV_M acts fully on M.*

Proof. Let $D := \{(x, y) \in A \times A : x = y\}$, we have by Theorem 3.4.2 that HV acts fully on D. Let $\phi_1 : M \to D$ and $\phi_2 : M^c \to D^c$ be bijections. Let $I_M : \Omega \to \Omega$ be the bijection defined by

$$(x)I_M = \left\{ \begin{array}{ll} (x)\phi_1 & x \in M \\ (x)\phi_2 & x \in M^c \end{array} \right\}.$$

As HV acts fully on D it follows that $I_M(HV)I_M^{-1}$ acts fully on M. So we also have that $I_MHI_M^{-1}I_MVI_M^{-1} = (I_MHI_M^{-1})(I_MVI_M^{-1})$ acts fully on M. Let $H_M := I_MHI_M^{-1}$ and $V_M := I_MVI_M^{-1}$. As these groups are conjugates of H and V, they are isomorphic to H and V. In particular they are abelian groups and we have the required result. □

Now that we have the required theorems and definitions, we will prove the main result of this section. In Lemma 5 of [14] Akos Seress makes use of a group D which contains elements of all possible disjoint cycle shapes, this idea is a critical part of the following proof in which we use a very similar group C.

Theorem 3.4.5. *The group* $\mathrm{Sym}(\Omega)$ *can be expressed as the product of 10 abelian groups.*

Proof. Let L be a moiety of A, $t \in A \backslash \{id_A\}$, $D := \{(x, y) \in A \times A : x = y\}$, $D_1 := \{(a, a) : a \in L\}$, $D_2 := \{(a, a+t) : a \in L\}$. Let $P := \{p_i : i \in |\Omega|\}$ be a partition of D_1^c into countable sets such that there are $|\Omega|$ sets of all cardinalities less than or equal to $\aleph_0$. For all $i \in |\Omega|$ let c_i be the group generated by a $|p_i|$-cycle on p_i. Let $C := \prod_{i \in |\Omega|} c_i$ (viewed as a

permutation group fixing all points of D_1). Finally let $H_{D_1}, V_{D_1^c}, H_{D_2^c}, V_{D_2^c}$ be as in Theorem 3.4.4.
Note that all the above groups are abelian as C is a product of cyclic groups acting on disjoint sets and the others are abelian by construction. We will show that $\mathrm{Sym}(\Omega) = HVHH_{D_1^c}V_{D_2^c}H_{D_1^c}V_{D_1^c}CV_{D_1^c}H_{D_1^c}$.
Let $g \in \mathrm{Sym}(\Omega)$ and $M := Dg^{-1}$. As the image of a moiety under a permutation we have that M is a moiety. By Theorem 3.4.3 we have that there is an element $g' \in HVH$ such that $Mg' \cap \{(a, a+t) : a \in A\} = \emptyset$.
Observe that $\{(a, a+t) : a \in L^c\} \subseteq (Mg')^c \backslash D_2$ and $|\{(a, a+t) : a \in L^c\}| = |\Omega| = |(\{(x, y) \in A \times A : x = y\} \cup D_2)^c|$. Let $\phi : (Mg')^c \backslash D_2 \to (D \cup D_2)^c$ be a bijection. Let $f \in \mathrm{Sym}(D_2^c)$ be defined by

$$(x)f = \left\{ \begin{array}{ll} (x)g'^{-1}g & x \in Mg' \\ (x)\phi & x \in (Mg')^c \end{array} \right\}.$$

As $H_{D_2^c}V_{D_2^c}$ acts fully on D_2^c there exists $g'' \in H_{D_2^c}V_{D_2^c}$ such that $g''|_{D_2^c} = f$. We now have that $(g'g'')|_M = g|_M$ and $g'g'' \in HVHH_{D_2^c}V_{D_2^c}$. It therefore follows that $s := (g'g'')^{-1}g \in \mathrm{Pstab}(D) = \mathrm{Sym}_\Omega(D^c) \leq_G \mathrm{Sym}_\Omega(D_1^c)$. As $s|_{D_1^c}$ fixes $D \backslash D_1$ it has $|\Omega|$ 1-cycles, therefore by the definition of C, by choosing either the identity or generating element of appropriately many c_i, there is an element $s' \in C$ such that $s'|_{D_1^c}$ has the same disjoint cycle shape as $s|_{D_1^c}$. As we have $s, s' \in \mathrm{Sym}_\Omega(D_L^c)$ have the same disjoint cycle shape when restricted to D_1^c there exists $w \in \mathrm{Sym}_\Omega(D_1^c)$ such that $ws'w^{-1} = s$. As $H_{D_1^c}V_{D_1^c}$ acts fully on D_1^c, we have that there exists $w' \in H_{D_1^c}V_{D_1^c}$ such that $w'|_{D_1^c} = w|_{D_1^c}$. For all $p \in \Omega$ we now have

$$(p)w's'w'^{-1} = \left\{ \begin{array}{ll} (p)w'w'^{-1} & p \in D_1 \\ (p)s & \text{otherwise} \end{array} \right\} = (p)s.$$

It follows that $s = w's'w'^{-1} \in H_{D_1^c}V_{D_1^c}CV_{D_1^c}H_{D_1^c}$. Therefore $g = g'g''(g'g'')^{-1}g = g'g''s \in HVHH_{D_2^c}V_{D_2^c}H_{D_1^c}V_{D_1^c}CV_{D_1^c}H_{D_L^c}$ as required. □

Chapter 4

Maximal Subgroups of Infinite Symmetric Groups

In this chapter we aim to construct large families of maximal subgroups of $\mathrm{Sym}(\Omega)$. In particular we will show that for any infinite set Ω there exists a family of $2^{2^{|\Omega|}}$ maximal subgroups of Ω which are pairwise non-conjugate. To do this we will first be building groups from ultrafilters using what was established in chapter 1.

4.1 Building Groups from Ultrafilters

The following Theorem will be useful when showing the maximality of certain subgroups.

Theorem 4.1.1. *Let Ω be an infinite set and let $G \leq_G \mathrm{Sym}(\Omega)$, then if all moieties of Ω are full in G then $G = \mathrm{Sym}(\Omega)$.*

Proof. The following proof is based on Note 3 of section 4 in [6].
Let G be a group such that all moieties of Ω are full in G.
Let M be a moiety of Ω. We have that M^c is also a moiety and therefore G acts on fully on M and M^c. Let $G_s := G \cap \mathrm{Sstab}(M)$. As G acts fully on M we have that G_s acts fully on M (as the required elements stabilise M setwise).
Let $\{M_1, M_2\}$ be a partition of M into moieties. Let $g \in \mathrm{Sym}_\Omega(M_1)$ be such that g has $|\Omega|$ cycles of all finite lengths. As $M_1 \cup M^c$ is a moiety there exists $g' \in G$ such that $g'|_{M_1 \cup M^c} = g|_{M_1 \cup M^c}$. We have that $g' \in G_s$ (as it fixes M^c pointwise).
We now construct a new element $g^* \in G_s$ by reversing all cycles of odd or infinite length of g' and preserving the others. This permutation is an element of G_s as it can be constructed by conjugating g' by an element of G_s with the required action on M (noting g' fixes M^c so it's action on M^c is unaffected by conjugation by elements of G_s).
In the product g^*g' all cycles of odd or infinite length cancel and all cycles of even length are squared. The square of a cycle of length $2n$ gives two cycles of length n. It follows that $g^*g' \in G_s$ has $2 * |\Omega| = |\Omega|$ cycles of all finite lengths and none of infinite length.
Therefore G_s contains an element of $\mathrm{Sym}_\Omega(M)$ which has $|\Omega|$ cycles of all finite lengths and none of infinite length. By conjugating this element by elements with the appropriate action on M we have that G_s contains all elements of $\mathrm{Sym}_\Omega(M)$ which have $|\Omega|$ cycles of all finite lengths and none of infinite length. By Theorem 2.3.7 $\mathrm{Sym}_\Omega(M) \subseteq G_s$. We now have that $\mathrm{Sym}_\Omega(M) \leq_G G$ for all moieties M and therefore by Theorem 3.1.3 we have $G = \mathrm{Sym}(\Omega)$. □

Definition 4.1.2 Given an ultrafilter $\mathcal{U}$ on an infinite set Ω.

$$F_\mathcal{U} := \{f \in \mathrm{Sym}(\Omega) : \mathrm{fix}(f) \in \mathcal{U}\}.$$

Theorem 4.1.3. *Given an ultrafilter $\mathcal{U}$ on an infinite set Ω, we have $F_\mathcal{U} \leq_G \mathrm{Sym}(\Omega)$.*

Proof. Let $f \in F_\mathcal{U}$ then $\mathrm{fix}(f) = \mathrm{fix}(f^{-1})$ so $f^{-1} \in F_\mathcal{U}$. We have that $F_\mathcal{U}$ is closed under inverses.
Let $f, g \in F_\mathcal{U}$ we have that $\mathrm{fix}(f)$ and $\mathrm{fix}(g)$ are in $\mathcal{U}$ therefore $\mathrm{fix}(f) \cap \mathrm{fix}(g) \in \mathcal{U}$. As $\mathrm{fix}(fg) \supseteq \mathrm{fix}(f) \cap \mathrm{fix}(g)$ we have $\mathrm{fix}(fg) \in \mathcal{U}$ and $fg \in F_\mathcal{U}$ so $F_\mathcal{U}$ is also closed under multiplication. □

Theorem 4.1.4. *Let $\mathcal{U}$ be an ultrafilter on an infinite set Ω, then $F_\mathcal{U}$ is transitive on the moieties of Ω in $\mathcal{U}$.*

Proof. The following proof is based on the proof of Theorem 6.4 in [5].
Let $M_1, M_2 \in \mathcal{U}$ be moieties of Ω. We will show that there is an element of $F_\mathcal{U}$ mapping M_1 to M_2.

Case 1: If $M_1 \backslash M_2$ and $M_2 \backslash M_1$ are both moieties then it follows that $|M_1 \backslash M_2| = |M_2 \backslash M_1|$. Let $\phi : M_1 \backslash M_2 \to M_2 \backslash M_1$ be a bijection and let $f \in \mathrm{Sym}(\Omega)$ be defined by

$$(x)f = \left\{ \begin{array}{lr} x & x \in M_1 \cap M_2 \\ x & x \in X \backslash (M_1 \cup M_2) \\ (x)\phi^{-1} & x \in M_2 \backslash M_1 \\ (x)\phi & x \in M_1 \backslash M_2 \end{array} \right\}.$$

As $M_1 \cap M_2 \subseteq \mathrm{fix}(f)$ it follows that $\mathrm{fix}(f) \in \mathcal{U}$ and therefore $f \in F_{\mathcal{U}}$. We have that $(M_1)f = M_2$ as required.

Case 2: Suppose that $M_1 \backslash M_2$ or $M_2 \backslash M_1$ is not a moiety. Without loss of generality we assume that $M_1 \backslash M_2$ is not a moiety. Then it follows that $|\Omega| = |(M_1 \backslash M_2)^c|$ and $|\Omega| > |M_1 \backslash M_2|$. We now have the following:

$$\begin{aligned} |M_1 \cap M_2| &= |M_1 \backslash (M_1 \backslash M_2)| = |M_1| - |M_1 \backslash M_2| = |\Omega|, \\ |(M_1 \cap M_2)^c| &= |M_1^c \cup M_2^c| = |\Omega|, \\ |M_1 \cup M_2| &= |\Omega|, \\ |(M_1 \cup M_2)^c| &= |(M_2 \cup (M_1 \backslash M_2))^c| = |M_2^c \cap (M_1 \backslash M_2)^c| = |M_2^c \backslash (M_1 \backslash M_2)| = |\Omega|. \end{aligned}$$

So both $M_1 \cap M_2$ and $M_1 \cup M_2$ are moieties. Let $\{M_3, M_4\}$ be a partition of $M_1 \cap M_2$ into moieties. By Theorem 1.4.5 either M_3 or M_3^c is in $\mathcal{U}$. If $M_3^c \in \mathcal{U}$ then $M_4 = M_3^c \cap M_1 \cap M_2 \in \mathcal{U}$. So we have that either M_3 or M_4 is in $\mathcal{U}$. Consider the set $M_5 := M_4 \cup (M_1 \cup M_2)^c$.

$M_5 \backslash M_1 = (M_1 \cup M_2)^c$ a moiety.
$M_1 \backslash M_5 = M_1 \backslash M_4$ a moiety (as it contains M_3 and its complement contains the complement of M_1).
$M_5 \backslash M_2 = (M_1 \cup M_2)^c$ a moiety.
$M_2 \backslash M_5 = M_2 \backslash M_4$ a moiety (as it contains M_3 and its complement contains the complement of M_2).

So by Case 1 there exist $f_1, f_2 \in F_{\mathcal{U}}$ such that $(M_1)f_1 = M_5$ and $(M_5)f_2 = M_2$, it follows that $(M_1)f_1f_2 = M_2$ and we have the required result. □

Theorem 4.1.5. *If $\mathcal{U}$ is an ultrafilter on an infinite set Ω, then $F_{\mathcal{U}}$ is transitive on the moieties of Ω not in $\mathcal{U}$.*

Proof. Let M_1 and M_2 be moieties not in $\mathcal{U}$ then both M_1^c and M_2^c are in $\mathcal{U}$ by Theorem 1.4.5. By Theorem 4.1.4 there is an $f \in F_{\mathcal{U}}$ such that $(M_1^c)f = M_2^c$. It therefore follows that $(M_1)f = M_2$ as required. □

Theorem 4.1.6. *If $\mathcal{U}$ is an ultrafilter on an infinite set Ω, then $F_{\mathcal{U}} \leq_G \mathrm{Sstab}(\mathcal{U})$.*

Proof. Let $f \in F_{\mathcal{U}}$ and let $U \in \mathcal{U}$. We have that $U \cap \mathrm{fix}(f) \in \mathcal{U}$ and $(U)f = (U \cap \mathrm{fix}(f)) \cup (U \backslash \mathrm{fix}(f))f \supseteq U \cap \mathrm{fix}(f)$ so we also have $(U)f \in \mathcal{U}$.
Similarly let $(U)f \in \mathcal{U}$. We have that $(U)f \cap \mathrm{fix}(f) \in \mathcal{U}$ and $U = ((U)f)f^{-1} = ((U)f \cap \mathrm{fix}(f)) \cup ((U)f \backslash \mathrm{fix}(f))f^{-1} \supseteq (U)f \cap \mathrm{fix}(f)$ so we also have $U \in \mathcal{U}$. □

Theorem 4.1.7. *If $\mathcal{U}$ is an ultrafilter on an infinite set Ω, then* $\mathrm{Sstab}(\mathcal{U}) = F_{\mathcal{U}}$.

Proof. The following proof is based on the proof of Theorem 6.4 in [5].
As by Theorem 4.1.6 we have $F_{\mathcal{U}} \leq_G \mathrm{Sstab}(\mathcal{U})$, it suffices to show that $\mathrm{Sstab}(\mathcal{U}) \leq_G F_{\mathcal{U}}$. Let $f \in \mathrm{Sstab}(\mathcal{U})$. We will construct two moieties M_1 and M_2 which partition Ω. Let $(c_i)_{i \in I}$ be the disjoint cycles of f (where I is an index set). Let each c_i of finite order $k > 1$ be given by $(c_{i,0}, c_{i,1} \ldots c_{i,k-1})$ and each c_i of infinite order be given by $(\ldots c_{i,-1}, c_{i,0}, c_{i,1} \ldots)$. Then we define M_1 and M_2 as follows:

1. If $|\mathrm{fix}(f)| < |\Omega|$ then $M_1 = \{c_{i,j} : i \in I \text{ and } j \text{ even }\} \cup \mathrm{fix}(f)$ and $M_2 = \{c_{i,j} : i \in I \text{ and } j \text{ odd }\}$. If there are $|\Omega|$ non-trivial cycles then each contributes at least one element to each of M_1 and M_2, thus $\{M_1, M_2\}$ is a partition of Ω into moieties. If not then as $|\mathrm{fix}(f)| < |\Omega|$ we have $|supp(f)| = |\Omega|$ so we must have $|\Omega| = \aleph_0$ and one of the cycles is infinite so $\{M_1, M_2\}$ is still a partition of Ω into moieties.

2. If $|\mathrm{fix}(f)| = |\Omega|$ then let $\{F_1, F_2\}$ be a partition of $\mathrm{fix}(f)$ into moieties. Let $M_1 := \{c_{i,j} : i \in I \text{ and } j \text{ even }\} \cup F_1$ and $M_2 = \{c_{i,j} : i \in I \text{ and } j \text{ odd }\} \cup F_2$. We have that $\{M_1, M_2\}$ is a partition of Ω into moieties.

Suppose for a contradiction that $f \notin F_{\mathcal{U}}$. It follows that $\mathrm{fix}(f) \notin \mathcal{U}$ and thus by Theorem 1.4.5 $supp(f) \in \mathcal{U}$. It follows from the definition of M_1 and M_2 that $supp(f) \subseteq M_1 \cup (M_1)f$ and similarly $supp(f) \subseteq M_2 \cup (M_2)f$ so we have that both of these sets are in $\mathcal{U}$.
By Theorem 1.4.5 we have that $(M_1 \cup (M_1)f)^c = M_2 \cap (M_2)f \notin \mathcal{U}$ and $(M_2 \cup (M_2)f)^c = M_1 \cap (M_1)f \notin \mathcal{U}$ so by the definition of a filter we have either M_1 or $(M_1)f$ is not in $\mathcal{U}$ and either M_2 or $(M_2)f$ not in $\mathcal{U}$. As $f \in \mathrm{Sstab}(\mathcal{U})$ it follows that none of $M_1, (M_1)f, M_2, (M_2)f$ are in $\mathcal{U}$ but as $M_1^c = M_2$ this contradicts Theorem 1.4.5. □

Theorem 4.1.8. *Let $\mathcal{U}$ be an ultrafilter on a set Ω and let g be a permutation of Ω. Then $\mathcal{V} := (\mathcal{U})g = \{(U)g : U \in \mathcal{U}\}$ is an ultrafilter on Ω.*

Proof. We first show that $\mathcal{V}$ is a filter.

1. As $\Omega \in \mathcal{U}$ we have that $\Omega = (\Omega)g \in \mathcal{V}$.

2. For $A, B \in \mathcal{V}$ we have $A = (A')g$ and $B = (B')g$ for some $A', B' \in \mathcal{U}$. As $\mathcal{U}$ is a filter it follows that $A' \cap B' \in \mathcal{U}$ and therefore $A \cap B = (A')g \cap (B')g = (A' \cap B')g \in \mathcal{V}$.

3. Let $A \in \mathcal{V}$ and let $B \supseteq A$. Then $A = (A')g$ for some $A' \in \mathcal{U}$ and $A' \subseteq Bg^{-1}$. So $Bg^{-1} \in \mathcal{U}$ and therefore $B \in \mathcal{V}$.

We now show that $\mathcal{V}$ is an ultrafilter. As $\emptyset \notin \mathcal{U}$ it follows that $\emptyset \notin \mathcal{V}$.
It therefore suffices to show that there is no filter $\mathcal{V}'$ such that $\mathcal{V} \subset \mathcal{V}' \subset P(\Omega)$. Suppose for a contradiction that there is such a $\mathcal{V}'$. Let $\mathcal{U}' := \{(V)g^{-1} : V \in \mathcal{V}'\}$. By using g^{-1} with the previous part of the proof we have that $\mathcal{U}'$ is a filter and we have that $\mathcal{U} \subset \mathcal{U}' \subset P(X)$. This contradicts the fact that $\mathcal{U}$ is an ultrafilter. ☐

Theorem 4.1.9. *Let $\mathcal{U}$ be an ultrafilter on an infinite set Ω. The group* $\mathrm{Sstab}(\mathcal{U})$ *is a maximal subgroup of* $\mathrm{Sym}(\Omega)$*.*

Proof. The following proof is based on the proof of Theorem 6.4 in [5].
First we show $\mathrm{Sstab}(\mathcal{U}) \neq \mathrm{Sym}(\Omega)$. Let $f \in \mathrm{Sym}(\Omega)$ be such that $\mathrm{fix}(f) = \emptyset$, by Theorem 4.1.7 it follows that $f \notin F_{\mathcal{U}} = \mathrm{Sstab}(\mathcal{U})$. Let $g \in \mathrm{Sym}(\Omega) \setminus \mathrm{Sstab}(\mathcal{U})$. Suppose for a contradiction that for all $M \in \mathcal{U}$ which are moieties of Ω we have $(M)g \in \mathcal{U}$. We have by Theorem 4.1.8 that $(\mathcal{U})g$ is an ultrafilter who's moieties are all contained in $\mathcal{U}$. In addition by Theorem 1.4.5 one of M and M^c is in $(\mathcal{U})g$ and $M^c \notin \mathcal{U}$ so $M \in (\mathcal{U})g$. It follows that $\mathcal{U}$ and $(\mathcal{U})g$ have the same moieties and therefore by Theorem 1.4.6 we have that $(\mathcal{U})g = \mathcal{U}$ a contradiction as $g \notin \mathrm{Sstab}(\mathcal{U})$. So we have that there is a moiety $M_1 \in \mathcal{U}$ such that $(M_1)g \notin \mathcal{U}$. Let M_2 be a moiety of Ω, by Theorem 1.4.5 precisely one of M_2 and M_2^c is in $\mathcal{U}$. Without loss of generality suppose that $M_2^c \in \mathcal{U}$. We have that $F_{\mathcal{U}}$ acts fully on M_2 as for all $h \in \mathrm{Sym}_{\Omega}(M_2)$ we have $\mathrm{fix}(h) \supseteq M_2^c \in \mathcal{U}$ and thus $\mathrm{fix}(h) \in \mathcal{U}$. By Theorem 4.1.7 $F_{\mathcal{U}} = \mathrm{Sstab}(\mathcal{U})$ so $\mathrm{Sstab}(\mathcal{U})$ acts fully on M_2.
By Theorem 4.1.5 for all $M \notin \mathcal{U}$ which are moieties of Ω there exists $h \in F_{\mathcal{U}} = \mathrm{Sstab}(\mathcal{U})$ such that $(M)h = M_2$ it follows that $\langle \mathrm{Sstab}(\mathcal{U}), g\rangle_G$ acts fully on all moieties of Ω not in $\mathcal{U}$.
By Theorems 4.1.4 and 4.1.5 for all $M \in \mathcal{U}$ which are moieties of Ω there exists $h_1, h_2 \in F_{\mathcal{U}} = \mathrm{Sstab}(\mathcal{U})$ such that $(M)h_1 = M_1$ and $((M_1)g)h_2 = M_2$. We now have that $(M)h_1gh_2 = M_2$ and therefore $\langle \mathrm{Sstab}(\mathcal{U}), g\rangle_G$ acts fully on all moieties of Ω in $\mathcal{U}$.
We now have that $\langle \mathrm{Sstab}(\mathcal{U}), g\rangle_G$ acts fully on all moieties of Ω and therefore by Theorem 4.1.1 $\langle \mathrm{Sstab}(\mathcal{U}), g\rangle_G = \mathrm{Sym}(\Omega)$. ☐

Theorem 4.1.10. *Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be distinct ultrafilters on an infinite set Ω. Then* $\mathrm{Sstab}(\mathcal{U}_1) \neq \mathrm{Sstab}(\mathcal{U}_2)$*.*

Proof. Let $U \in \mathcal{U}_1 \setminus \mathcal{U}_2$ be a moiety (this must exist by Theorem 1.4.6). By Theorem 4.1.7 it suffices to prove $F_{\mathcal{U}_1} \neq F_{\mathcal{U}_2}$. Choose $f \in \mathrm{Sym}(\Omega)$ such that $\mathrm{fix}(f) = U$ then $f \in F_{\mathcal{U}_1} \setminus F_{\mathcal{U}_2}$ and therefore $F_{\mathcal{U}_1} \neq F_{\mathcal{U}_2}$ as required. ☐

Theorem 4.1.11. *For all infinite sets Ω, there exists a family of $2^{2^{|\Omega|}}$ pairwise non-conjugate maximal subgroups of* $\mathrm{Sym}(\Omega)$*.*

Proof. The following proof is based on the proof of corollary 6.5 in [5].
By Theorem 1.4.11 there are $2^{2^{|\Omega|}}$ ultrafilters on Ω. It follows from Theorems 4.1.10 and 4.1.9 that there exists a family of $2^{2^{|\Omega|}}$ maximal subgroups of $\mathrm{Sym}(\Omega)$. As there are only $2^{|\Omega|}$ elements of $\mathrm{Sym}(\Omega)$ it follows that this family can be partitioned into $2^{2^{|\Omega|}}$ conjugacy classes. By choosing one element of each we have the required result. ☐

4.2 Finite Partition Stabilisers

In this section we will explore more examples of maximal subgroups of infinite symmetric groups in the form of partition stabilisers.

Theorem 4.2.1. *Let Ω be an infinite set, and let $F \subseteq \Omega$ be finite. We then have that* $\mathrm{Sstab}(\{F, F^c\})$ *is a maximal subgroup of* $\mathrm{Sym}(\Omega)$*.*

Proof. Let $f \in \mathrm{Sym}(\Omega) \setminus \mathrm{Sstab}(F)$ and $g \in \mathrm{Sym}(\Omega)$. As $\mathrm{Sstab}(\{F, F^c\}) = \mathrm{Sstab}(F)$ it suffices to show that $g \in \langle \mathrm{Sstab}(F), f\rangle_G$.
As $f \notin \mathrm{Sstab}(F)$ there must exist a point $p \in F$ such that $(p)f \in F^c$. Let $p_2 \in (F^c)f \cap F^c$. We have that $((p)f, p_2) \in \mathrm{Sstab}(F)$. It therefore follows that $(p, (p_2)f^{-1}) = f((p)f, p_2)f^{-1} \in \langle \mathrm{Sstab}(F), f\rangle_G$. For all $a \in F$ and $b \in F^c$ we have that $(a, b) = (a, p)(p, (p_2)f^{-1})((p_2)f^{-1}, b) \in \langle \mathrm{Sstab}(F), f\rangle_G$. Let F be indexed by $F = \{p_i : i < k\}$. We now have that $g(p_1, (p_1)g)(p_2, (p_2)g) \dots (p_k, (p_k)g) \in \mathrm{Sstab}(F)$. It therefore follows that $g \in \langle \mathrm{Sstab}(F), f\rangle_G$ as required. ☐

Theorem 4.2.2. *Let Ω be an infinite set and let Σ_1 and Σ_2 be infinite subsets of Ω such that $|\Sigma_1 \cap \Sigma_2| = |\Sigma_1 \cup \Sigma_2|$. We have that $\mathrm{Sym}_\Omega(\Sigma_1 \cup \Sigma_2) = \langle \mathrm{Sym}_\Omega(\Sigma_1), \mathrm{Sym}_\Omega(\Sigma_2)\rangle_G$.*

Proof. The following proof is based on the proof of the first lemma of [9].
Let $f \in \mathrm{Sym}_\Omega(\Sigma_1 \cup \Sigma_2)$. We have that either $|(\Sigma_1 \cap \Sigma_2)f \cap \Sigma_1| = |\Sigma_1 \cap \Sigma_2|$ or $|(\Sigma_1 \cap \Sigma_2)f \cap \Sigma_2| = |\Sigma_1 \cap \Sigma_2|$. Without loss of generality we assume that $|(\Sigma_1 \cap \Sigma_2)f \cap \Sigma_1| = |\Sigma_1 \cap \Sigma_2|$. Let M be a moiety of $(\Sigma_1 \cap \Sigma_2)f \cap \Sigma_1$. It follows that M is a moiety of Σ_1 and $(M)f^{-1}$ is a moiety of $\Sigma_1 \cap \Sigma_2$, Σ_1 and Σ_2. Let $f' \in \mathrm{Sym}_\Omega(\Sigma_1)$ be such that $(p)ff' = p$ for all $p \in (M)f^{-1}$. As $|\Sigma_1 \cap \Sigma_2| = |\Sigma_1|$ we have that $\Sigma_1 \backslash \Sigma_2$ is contained in a moiety of Σ_1 and thus there exists an element $g \in \mathrm{Sym}_\Omega(\Sigma_1)$ such that $\Sigma_1 \backslash \Sigma_2 \subseteq Mf^{-1}g$. As ff' fixes Mf^{-1} it follows that $g^{-1}ff'g \in \mathrm{Sym}_\Omega(\Sigma_1 \cup \Sigma_2)$ fixes $\Sigma_1 \backslash \Sigma_2$ and thus $g^{-1}ff'g \in \mathrm{Sym}_\Omega(\Sigma_2)$. So we have that $f \in g\,\mathrm{Sym}_\Omega(\Sigma_2)g^{-1}f'^{-1} \subseteq \langle \mathrm{Sym}_\Omega(\Sigma_1), \mathrm{Sym}_\Omega(\Sigma_2)\rangle_G$ as required. □

Definition 4.2.3 Let P be a partition of an infinite set Ω into finitely many sets and let κ be an infinite cardinal. The κ-almost stabiliser of P is defined by

$$\mathrm{Sstab}(P)_{<\kappa} := \{f \in \mathrm{Sym}(\Omega) : \text{there exists } f' \in \mathrm{Sym}(P) \text{ such that for all } p \in P \text{ we have } |(p)f \triangle (p)f'| < \kappa \text{ and } |p| = |(p)f'|\}.$$

Theorem 4.2.4. *Let P be a partition of an infinite set Ω into finitely many sets and let κ be an infinite cardinal. Then the κ-almost stabiliser of P is a group.*

Proof. Let $f, g \in \mathrm{Sstab}(P)_{<\Omega}$ and let $p \in P$. As $(p)fg \triangle (p)f'g' \subseteq (((p)f \triangle (p)f')g \cup ((p)f'g \triangle pf'g'))$ we have that $|(p)fg \triangle (p)f'g'| \leq |((p)f \triangle (p)f')g| + |((p)f')g \triangle (p)f'g')| < \kappa + \kappa = \kappa$ and thus $fg \in \mathrm{Sstab}(P)_{<\kappa}$.
It now suffices to show that $f^{-1} \in \mathrm{Sstab}(P)_{<\kappa}$. Let $p \in P$. We have that $|(p)f'^{-1}| = |p|$ and

$$|(p)f^{-1} \triangle (p)f'^{-1}| = |(p)f^{-1}f \triangle (p)f'^{-1}f| = |(p) \triangle (p)f'^{-1}f| = |((p)f'^{-1})f' \triangle ((p)f'^{-1})f| < \kappa.$$

□

Theorem 4.2.5. *Let Ω be an infinite set, and let $P := \{M_0, M_1 \ldots M_k\}$ for some $k \geq 1$ be a partition of Ω into finitely many moieties. We then have that $\mathrm{Sstab}(P)$ is not a maximal subgroup of $\mathrm{Sym}(\Omega)$. In particular $\mathrm{Sstab}(P) <_G \mathrm{Sstab}(P)_{<|\Omega|}$ which is maximal.*

Proof. The following proof is based on the proof of observation 6.2 in [5].
Let $x_0 \in M_0$ and $x_1 \in M_1$ we have that $(x_0, x_1) \in \mathrm{Sstab}(P)_{<|\Omega|} \backslash \mathrm{Sstab}(P)$ and thus $\mathrm{Sstab}(P) <_G \mathrm{Sstab}(P)_{<|\Omega|}$.
We now show that $\mathrm{Sstab}(P)_{<|\Omega|}$ is maximal. We first show that $\mathrm{Sstab}(P)_{<|\Omega|} \neq \mathrm{Sym}(\Omega)$. Let $\{M_{0,1}, M_{0,2}\}$ be a partition of M_0 into moieties of M_0 and $\{M_{1,1}, M_{1,2}\}$ be a partition of M_1 into moieties of M_1. Then we have that $|M_{0,1}| = |M_{1,1}|$ and thus there is a bijection $\phi : M_{0,1} \to M_{1,1}$. Let $g \in \mathrm{Sym}(\Omega)$ be defined by

$$(x)g = \left\{ \begin{array}{ll} (x)\phi & x \in M_{0,1} \\ (x)\phi^{-1} & x \in M_{1,1} \\ x & \text{otherwise} \end{array} \right\}.$$

We have that $g \notin \mathrm{Sstab}(P)_{<|\Omega|}$ as $|(M_0)g \triangle M_i| = |\Omega|$ for all $i \leq k$.
Let $f \in \mathrm{Sym}(\Omega) \backslash \mathrm{Sstab}(P)_{<|\Omega|}$. It suffices to show that $\langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G = \mathrm{Sym}(\Omega)$.

Claim: There exist i, j_1, j_2 such that $j_1 \neq j_2$, $|(M_i)f \cap M_{j_1}| = |\Omega|$ and $|(M_i)f \cap M_{j_2}| = |\Omega|$.
Proof of Claim: Suppose for a contradiction that our Claim is false. We partition each M_i as $\{M_{i,0}, M_{i,1} \ldots M_{i,k}\}$ where $M_{i,j} := \{x \in M_i : (x)f \in M_j\}$. As each M_i is a moiety we have that for every i, that at least one of $M_{i,0}, M_{i,1} \ldots M_{i,k}$ is a moiety. We also have at most one of these is a moiety as if not that would contradict our assumption that the Claim is false. In addition for each j there must be an i such that $M_{i,j}$ is a moiety, and $|M_j \backslash (M_{i,j})f| < |\Omega|$ as if this were not the case then f would not be onto M_j. By letting $f' \in \mathrm{Sym}(P)$ be defined by $(M_i)f' = M_j$ for i, j such that $M_{i,j}$ is a moiety, it follows that $f \in \mathrm{Sstab}(P)_{<|\Omega|}$ a contradiction. □
We now have

$$\mathrm{Sym}_\Omega((M_i)f \cap (M_{j_1} \cup M_{j_2})) \leq_G f^{-1}\,\mathrm{Sym}_\Omega(M_i)f \leq_G \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G,$$

$$\mathrm{Sym}_\Omega(M_{j_1}) \leq_G \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G, \qquad \mathrm{Sym}_\Omega(M_{j_2}) \leq_G \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G.$$

It follows by Theorem 4.2.2 that $\mathrm{Sym}_\Omega(M_{j_1} \cup ((M_i)f \cap (M_{j_1} \cup M_{j_2}))) \leq_G \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G$. By using Theorem 4.2.2 again we then have that $\mathrm{Sym}_\Omega((M_{j_2} \cup M_{j_1} \cup ((M_i)f \cap (M_{j_1} \cup M_{j_2})))) = \mathrm{Sym}_\Omega(M_{j_1} \cup M_{j_2}) \leq_G \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G$.
As the elements of P all have the same cardinality we can permute them using elements of $\mathrm{Sstab}(P)$. If follows that for all $i, j \leq k$ we have $\mathrm{Sym}_\Omega(M_i \cup M_j) \leq_G \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G$. By repeatedly applying Theorem 4.2.2 we will get the required

result as follows

$$\begin{aligned}
&\mathrm{Sym}_\Omega(M_0 \cup M_1) \leq_G \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G \text{ and } \mathrm{Sym}_\Omega(M_1 \cup M_2) \leq_G \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G \\
&\implies \mathrm{Sym}_\Omega(M_0 \cup M_1 \cup M_2) \leq_G \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G \text{ and } \mathrm{Sym}_\Omega(M_2 \cup M_3) \leq_G \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G \\
&\implies \mathrm{Sym}_\Omega(M_0 \cup M_1 \cup M_2 \cup M_3) \leq_G \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G \text{ and } \mathrm{Sym}_\Omega(M_3 \cup M_4) \leq_G \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G \\
&\quad\vdots \\
&\implies \mathrm{Sym}_\Omega(M_0 \cup M_1 \ldots \cup M_k) \leq_G \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G \\
&\implies \mathrm{Sym}_\Omega(\cup P) \leq_G \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G \implies \mathrm{Sym}(\Omega) \leq_G \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G.
\end{aligned}$$

□

Theorem 4.2.6. *Let Ω be an uncountable set, and let $P := \{N, N^c\}$ be a partition of Ω where $\aleph_0 \leq |N| < |\Omega|$. We then have that $\mathrm{Sstab}(P)$ is not a maximal subgroup of $\mathrm{Sym}(\Omega)$. In particular $\mathrm{Sstab}(P) <_G \mathrm{Sstab}(P)_{<|N|}$ which is maximal.*

Proof. Let $x_1 \in N$ and $x_2 \in N^c$ we have that $(x_1, x_2) \in \mathrm{Sstab}(P)_{<|N|} \backslash Sstab(P)$ and thus $\mathrm{Sstab}(P) < \mathrm{Sstab}(P)_{<|N|}$. We now show that $\mathrm{Sstab}(P)_{<|N|}$ is maximal. We first show that $\mathrm{Sstab}(P)_{<|N|} \neq \mathrm{Sym}(\Omega)$. Let $N' \subseteq N^c$ be such that $|N| = |N'|$. Then we have that there is a bijection $\phi : N \to N'$. Let h be defined by

$$(x)h = \left\{ \begin{array}{ll} (x)\phi & x \in N \\ (x)\phi^{-1} & x \in N' \\ x & \text{otherwise} \end{array} \right\}.$$

We have that $h \notin \mathrm{Sstab}(P)_{<|N|}$ as $|(N)h \triangle N| = |N|$ and $|(N)h \triangle N^c| = |\Omega|$. Let $f \in \mathrm{Sym}(\Omega) \backslash \mathrm{Sstab}(P)_{<|N|}$, it suffices to show that $\langle f, \mathrm{Sstab}(P)_{<|N|}\rangle_G = \mathrm{Sym}(\Omega)$.
Claim: Either $|(N)f \cap N^c| = |N|$ or $|(N)f^{-1} \cap N^c| = |N|$.
Proof of Claim: As $f \notin \mathrm{Sstab}(P)_{<|N|}$ we have that either $|N \triangle (N)f| = |N|$ or $|N^c \triangle (N^c)f| = |N|$. We therefore have one of the following:

1. If $|N \backslash (N)f| = |N|$ then $|N| = |(N)f^{-1} \backslash N| = |(N)f^{-1} \cap N^c|$.
2. If $|(N)f \backslash N| = |N|$ then $|(N)f \cap N^c| = |N|$.
3. If $|N^c \backslash (N^c)f| = |N|$ then $|N| = |(N^c)f^{-1} \backslash N^c| = |(N^c)f^{-1} \cap N| = |N^c \cap (N)f|$.
4. If $|(N^c)f \backslash N^c| = |N|$ then $|N| = |(N^c)f \cap N| = |N^c \cap (N)f^{-1}|$. □

As $\langle f, \mathrm{Sstab}(P)_{<|N|}\rangle_G = \langle f^{-1}, \mathrm{Sstab}(P)_{<|N|}\rangle_G$ we may assume without loss of generality that $|Nf \cap N^c| = |N|$. Let $\{N_0, N_1\}$ be a partition of $Nf \cap N^c$ into moieties. Let $N_2 \subset N^c$ be such that $N_2 \cap Nf \cap N^c = \emptyset$ and $|N_2| = |N|$. Let $\phi : N_1 \to N_2$ be a bijection and let $h \in \mathrm{Sstab}(P)$ be defined by

$$(x)h = \left\{ \begin{array}{ll} (x)\phi & x \in N_1 \\ (x)\phi^{-1} & x \in N_2 \\ x & \text{otherwise} \end{array} \right\}.$$

It follows that $(N_1)h = N_2$ and thus $((N_1)f^{-1})fhf^{-1} = (N_2)f^{-1} \subset N^c$. As $(N_1)f^{-1}$ is a moiety of N we have shown that there is an involution in $\langle f, \mathrm{Sstab}(P)_{<|N|}\rangle_G$ which swaps a moiety of N with a subset of N^c and fixes all other points. It follows that, by conjugating this involution by elements of $\mathrm{Sstab}(P)$, we can construct an involution which swaps any moiety of N with any subset of N^c with cardinality $|N|$ and fixes all other points. By partitioning N and a subset of N^c into moieties we can therefore construct an involution in $\langle f, \mathrm{Sstab}(P)_{<|N|}\rangle_G$ which swaps N with any subset of N^c with cardinality $|N|$ and fixes all other points. Let M be a moiety of Ω and let $N_M \subseteq M^c \backslash N$ be such that $|N_M| = |N|$. Let $g \in \langle f, \mathrm{Sstab}(P)_{<|N|}\rangle_G$ be an involution swapping N_M with N and fixing all other points. We have that $g\,\mathrm{Sstab}(P)g^{-1}$ acts fully on M and thus as M was arbitrary we have $\langle f, \mathrm{Sstab}(P)_{<|N|}\rangle_G$ acts fully on all moieties of Ω. We therefore have $\langle f, \mathrm{Sstab}(P)_{<|N|}\rangle_G = \mathrm{Sym}(\Omega)$ by Theorem 4.1.1. □

We now have all the theorems required to classify which finite partition stabilisers yield maximal subgroups.

Theorem 4.2.7. *Let Ω be an infinite set and let $P := \{\Omega_0, \Omega_1 \ldots \Omega_k\}$ be a partition of Ω into finitely many sets. Then $\mathrm{Sstab}(P)$ is a maximal subgroup of $\mathrm{Sym}(\Omega)$ if and only if $P = \{F, F^c\}$ where F is finite or $P = \{S_1, S_2 \ldots S_{k-1}, \left(\bigcup_{i<k} S_i\right)^c\}$ where $S_1, S_2 \ldots S_{k-1}$ are singletons.*

Proof. We will consider all the ways of partitioning Ω into finitely many sets.

1. Suppose that $P = \{S_0, S_1 \dots S_{k-1}, (\cup_{i<k} S_i)^c\}$ where $S_0, S_1 \dots S_{k-1}$ are singletons.
   It follows that $\text{Sstab}(P) = \text{Sstab}(\{\cup_{i<k} S_i, (\cup_{i<k} S_i)^c\})$ and therefore by Theorem 4.2.1 that $\text{Sstab}(P)$ is maximal.

2. Suppose that $P = \{S_0, S_1 \dots S_{k_1}, \Omega_0, \Omega_1 \dots \Omega_{k_2}\}$ where $S_0, S_1 \dots S_{k_1}$ are singletons, $\Omega_0, \Omega_1 \dots \Omega_{k_2}$ are not singletons, $k_1, k_2 \in \mathbb{N}$ and $k_2 \geq 2$. Let $x_1 \in \Omega_1$ and $x_2 \in \Omega_2$. We have that $(x_1, x_2) \in \text{Sstab}(\{\cup_{i \leq k_1} S_i, \cup_{i \leq k_2} \Omega_i\}) \setminus \text{Sstab}(P)$ and therefore we have $\text{Sstab}(P) <_G \text{Sstab}(\{\cup_{i \leq k_1} S_i, \cup_{i \leq k_2} \Omega_i\})$ which by Theorem 4.2.1 is maximal and therefore $\text{Sstab}(P)$ is not maximal.

3. Suppose that $P = \{F_0, F_1 \dots F_{k_1}, \Omega_0, \Omega_1 \dots \Omega_{k_2}\}$ where $F_1, F_2 \dots F_{k_1}$ are finite sets at least one of which has at least 2 elements, $\Omega_0, \Omega_1 \dots \Omega_{k_2}$ are infinite $k_1, k_2 \in \mathbb{N}$ and at least one of k_1, k_2 is not 1. Similarly to case 2 if $k_2 \neq 1$ then $\text{Sstab}(P) <_G \text{Sstab}(\{\cup_{i \leq k_1} F_i, \cup_{i \leq k_2} \Omega_i\})$ which by Theorem 4.2.1 is maximal and therefore $\text{Sstab}(P)$ is not maximal. If $k_2 = 1$ then we have without loss of generality that F_0 has at least 2 elements and $k_1 > 1$. Therefore if $x_1 \in F_0$ and $x_2 \in F_1$ then we have that $(x_1, x_2) \in \text{Sstab}(\{\cup_{i \leq k_1} F_i, \cup_{i \leq k_2} \Omega_i\}) \setminus \text{Sstab}(P)$ and so $\text{Sstab}(P) <_G \text{Sstab}(\{\cup_{i \leq k_1} F_i, \cup_{i \leq k_2} \Omega_i\})$ which by Theorem 4.2.1 is maximal and therefore $\text{Sstab}(P)$ is not maximal.

4. Suppose that $P = \{F, F^c\}$ where F is finite. It follows immediately by Theorem 4.2.1 that $\text{Sstab}(P)$ is maximal.

5. Suppose that $P = \{N, \Omega_0, \Omega_1 \dots \Omega_k\}$ where $k \in \mathbb{N}$ and $\aleph_0 \leq |N| < |\Omega|$. We have that $\text{Sstab}(P) \leq_G \text{Sstab}(\{\cup\{s \in P : |s| = |N|\}, \{\cup\{s \in P : |s| \neq |N|\}\}\})$ which is not maximal by Theorem 4.2.6 and therefore $\text{Sstab}(P)$ is not maximal.

6. Suppose that $P = \{M_1, M_2 \dots M_k\}$ a partition into moieties of Ω. It follows immediately from Theorem 4.2.5 that $\text{Sstab}(P)$ is not maximal.

□

Chapter 5

Conclusion

I have now proved all the results mentioned in the introduction. If I were to continue, there are various topics I could explore. Such as the stabilisers of infinite partitions, topologies on the symmetric groups of uncountable sets, or I could try and decrease the bound from the abelian product section further. I have thoroughly enjoyed learning about infinite symmetric groups and working on my project as a whole.

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